

THE CONTINUUM RANDOM TREE IS THE SCALING LIMIT OF UNLABELLED UNROOTED TREES

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ABSTRACT. We prove that the uniform unlabelled unrooted tree with n vertices and vertex degrees in a fixed set converges in the Gromov-Hausdorff sense after a suitable rescaling to the Brownian continuum random tree. This proves a conjecture by Aldous. Moreover, we establish Benjamini-Schramm convergence of this model of random trees.

1. INTRODUCTION AND MAIN RESULTS

Since the construction of the continuum random tree (CRT) by Aldous [Ald91b, Ald91c, Ald93] various models of random structures have been found to admit the CRT as scaling limit, see for example Albenque and Marckert [AM08], Bettinelli [Bet15], Caraceni [Car], Curien, Haas and Kortchemski [CHK14], Haas and Miermont [HM12], Marckert and Miermont [MM11], Janson and Stefansson [JS15], or Panagiotou, Stuffer, and Weller [PSW14]. Moreover, the CRT has incited research in many other directions, among we mention the recent advances by Abraham and Le Gall [AL15], and Albenque and Goldschmidt [AG15].

The present paper concerns itself with trees that are unordered and unlabelled. Here one distinguishes between Pólya trees, which have a root, and unlabelled (unrooted) trees. It has been a long-standing conjecture by Aldous [Ald91c, p. 55] that the models "all Pólya trees with n -vertices equally likely" and "all unlabelled trees with n -vertices equally likely" admit the CRT as scaling limit. The convergence of random Pólya trees has been confirmed by Haas and Miermont [HM12], and an alternative proof has been given later by Panagiotou and Stuffer [PS15]. As was stated explicitly in [HM12], this does not settle the question regarding the convergence of random unlabelled trees. Our first main result confirms the CRT as scaling limit of unlabelled unrooted trees. We take a unified approach to cover all (sensible) cases of vertex degree restrictions.

Theorem 1.1. *Let Ω be a set of positive integers containing 1 and at least one integer equal or larger than 3, and set $\Omega^* = \Omega - 1$. Let T_n be drawn uniformly at random from the unlabelled trees with n vertices and vertex-degrees in Ω . Then there is a constant e_Ω such that*

$$(1.1) \quad (\mathsf{T}_n, e_\Omega n^{-1/2} d_{\mathsf{T}_n}) \xrightarrow{(d)} (\mathcal{T}_e, d_{\mathcal{T}_e})$$

in the Gromov-Hausdorff sense, as $n \equiv 2 \pmod{\gcd(\Omega^)}$ becomes large. Moreover, there are constants $C, c > 0$ such that the diameter $D(\mathsf{T}_n)$ satisfies the tail bound*

$$(1.2) \quad \mathbb{P}(D(\mathsf{T}_n) \geq x) \leq C \exp(-cx^2/n)$$

for all $x \geq 0$.

Similar bounds were obtained by Addario-Berry, Devroye, and Janson [ABDJ13] for the height of random critical Galton-Watson trees conditioned to be large, if the offspring distribution has finite nonzero variance. Our results show that (1.2) is optimal up to the choice of the constants C, c .

Although scaling limits describe asymptotic global properties, they do not contain information on local properties, such as the limiting degree distribution of a randomly chosen vertex in a graph. Such asymptotic local properties of random rooted structures are described by Benjamini-Schramm limits. Here convergence means convergence of neighbourhoods around randomly drawn roots toward neighbourhoods around the root of a random limit graph. Our second main result establishes Benjamini-Schramm convergence for random unlabelled unrooted trees toward an infinite limit tree. We take a unified approach to cover all sensible cases of vertex degree restrictions.

Theorem 1.2. *Let Ω be a set of positive integers containing 1 and at least one integer equal or larger than 3, and set $\Omega^* = \Omega - 1$. Let T_n be drawn uniformly at random from the unlabelled trees with n vertices and vertex-degrees in Ω . Then T_n converges in the Benjamini Schramm sense toward an infinite rooted tree (A_Ω, u) , as $n \equiv 2 \pmod{\gcd(\Omega^*)}$ becomes large.*

The limit tree (A_Ω, u) is identical to the Benjamini-Schramm limit of random Pólya trees with vertex outdegrees in the shifted set Ω^* , which was established in [Stu15] using results of Aldous [Ald91a]. The tree is constructed by starting with a modified version of Kesten's tree (that is, the family tree of a Galton-Watson process conditioned to survive), in which only the distribution of the root degree may differ from the original Kesten tree, and attaching random trees to each of its vertices. The root u is drawn uniformly at random from the tree attached to the root of the modified Kesten tree.

There is a connection to the local weak limit (B_Ω, o) of random Pólya trees with vertex outdegrees in Ω^* in [Stu15], that is, with respect to convergence of neighbourhoods of the fixed roots of these trees. The trees A_Ω and B_Ω are similar (roughly speaking they differ only in the same way as the Kesten tree differs from its modified version), and are in fact identically distributed in the case $\Omega = \mathbb{N}$. Only the root o is always fixed, as opposed to the randomly drawn root u .

While the convergence of random unlabelled unrooted trees with proper vertex degree restrictions is new, the convergence of random unlabelled unrooted trees without vertex degree restrictions, that is, the case $\Omega = \mathbb{N}$, was independently obtained by Georgakopoulos and Wagner in their recent work [GW]. They applied a direct combinatorial approach that does not build on the Benjamini-Schramm limit of random rooted trees, and obtained a different description of the limit object, that does not use the connection to branching processes.

Our methods for the proof of Theorems 1.1 and 1.2 are based on the cycle pointing decomposition developed by Bodirsky, Fusy, Kang and Vigerske [BFKV11]. This allows us to approximate the random unlabelled unrooted tree T_n with n vertices and vertex outdegrees in a set Ω , by random Pólya trees with vertex outdegrees in the shifted set $\Omega^* = \Omega - 1$, whose random sizes concentrate around n . The approximation works well with respect to both local and global properties. We note that our arguments apparently do not work in the other direction. That is, the convergence of T_n (in the Benjamini-Schramm sense, or in the sense of scaling limits) could be used to obtain convergence of a random Pólya-tree having a *random* number of vertices, but, although this number concentrates, this is not sufficient to deduce convergence of a random Pólya tree with a fixed size.

A direct consequence of the scaling limit is that the rescaled diameter $e_\Omega n^{-1/2} D(\mathsf{T}_n)$ converges weakly and in arbitrarily high moments toward the diameter $D(\mathcal{T}_e)$ of the CRT. That is,

$$\mathbb{P}(n^{-1/2} e_\Omega D(\mathsf{T}_n) > x) \rightarrow \mathbb{P}(D(\mathcal{T}_e) > x),$$

and

$$\mathbb{E}[D(\mathsf{T}_n)^p] \sim e_\Omega^{-p} n^{p/2} \mathbb{E}[D(\mathcal{T}_e)^p].$$

The distribution of $D(\mathcal{T}_e)$ is known and given by

$$(1.3) \quad D(\mathcal{T}_e) \stackrel{(d)}{=} \sup_{0 \leq t_1 \leq t_2 \leq 1} (\mathbf{e}(t_1) + \mathbf{e}(t_2) - 2 \inf_{t_1 \leq t \leq t_2} \mathbf{e}(t)),$$

with $\mathbf{e} = (\mathbf{e}_t)_{0 \leq t \leq 1}$ denoting Brownian excursion of length 1, and

$$(1.4) \quad \mathbb{P}(D(\mathcal{T}_e) > x) = \sum_{k=1}^{\infty} (k^2 - 1) \left(\frac{2}{3} k^4 x^4 - 4k^2 x^2 + 2 \right) \exp(-k^2 x^2 / 2).$$

Equations (1.3) and (1.4) the first moment $\mathbb{E}[D(\mathcal{T}_e)] = 4/3\sqrt{\pi/2}$ have been known since the construction of the CRT by Aldous [Ald91c, Ch. 3.4], who used the convergence of random labelled trees to the CRT together with results by Szekeres [Sze83] regarding the diameter of these trees. Expression (1.4) was recently recovered directly in the continuous setting by Wang [Wan15].

The (known) moments of the diameter are given by:

$$(1.5) \quad \mathbb{E}[D(\mathcal{T}_e)] = \frac{4}{3}\sqrt{\pi/2}, \quad \mathbb{E}[D(\mathcal{T}_e)^2] = \frac{2}{3} \left(1 + \frac{\pi^2}{3} \right), \quad \mathbb{E}[D(\mathcal{T}_e)^3] = 2\sqrt{2\pi},$$

$$(1.6) \quad \mathbb{E}[D(\mathcal{T}_e)^k] = \frac{2^{k/2}}{3} k(k-1)(k-3)\Gamma(k/2)(\zeta(k-2) - \zeta(k)) \quad \text{for } k \geq 4.$$

The expression $\mathbb{E}[D(\mathcal{T}_e)] = \frac{4}{3}\sqrt{\pi/2}$ may be obtained as shown in Aldous [Ald91c, Sec. 3.4] using results of Szekeres [Sze83], who proved the existence of a limit distribution for the diameter of rescaled random unordered labelled trees. The higher moments could be obtained in the same way by elaborated calculations, or, we can deduce them by building on results by Broutin and Flajolet, who studied in [BF12] the random tree τ_n that is drawn uniformly at random among all unlabelled trees with n leaves in which each inner vertex is required to have degree 3. Using analytic methods [BF12, Thm. 8], they computed asymptotics of the form

$$\mathbb{E}[D(\tau_n)^r] \sim c_r \lambda^{-r} n^{r/2}$$

with λ an analytically given constant, and the constants c_r given by

$$\begin{aligned} c_1 &= \frac{8}{3}\sqrt{\pi}, \quad c_2 = \frac{16}{3}(1 + \frac{\pi^2}{3}), \quad c_3 = 64\sqrt{\pi}, \\ c_r &= \frac{4^r}{3} r(r-1)(r-3)\Gamma(r/2)(\zeta(r-2) - \zeta(r)) \quad \text{if } r \geq 4. \end{aligned}$$

As τ_n has n leaves and hence $2n-1$ vertices in total, it follows by Theorem 1.1 that

$$(\tau_n, e_{\{0,2\}}(2n-1)^{-1/2} d_{\tau_n}) \xrightarrow{(d)} (\mathcal{T}_e, d_{\mathcal{T}_e})$$

and consequently, by the exponential tail-bounds for the diameter which imply arbitrarily high uniform integrability,

$$\mathbb{E}[D(\tau_n)^r] \sim \mathbb{E}[D(\mathcal{T}_e)^r] (e_{\{0,2\}}/\sqrt{2})^{-r} n^{r/2}.$$

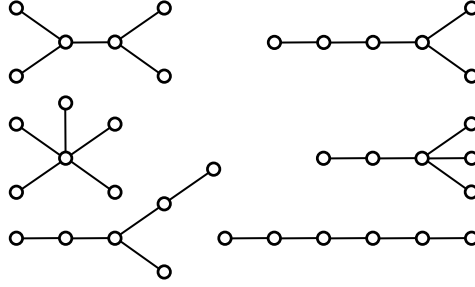


Figure 1. All unlabelled (unrooted) trees with 6 vertices.

It follows that $\mathbb{E}[D(\mathcal{T}_e)^r] = c_r(e_{\{0,2\}}/(\sqrt{2}\lambda))^r$. All that remains is to calculate the ratio $e_{\{0,2\}}/(\sqrt{2}\lambda)$, which is given by

$$e_{\{0,2\}}/(\sqrt{2}\lambda) = \mathbb{E}[D(\mathcal{T}_e)]/c_1 = 1/(2\sqrt{2}),$$

since $\mathbb{E}[D(\mathcal{T}_e)] = 4/3\sqrt{\pi/2}$. This yields Equations (1.5) and (1.6).

Outline. In Section 2 we fix basic graph theoretic notions, in particular regarding unlabelled trees. Section 3 gives a brief account on Gromov-Hausdorff convergence and the continuum random tree. Section 4 recalls the notion of local weak convergence and results for random Pólya trees. Section 5 introduces the reader to the notion of combinatorial species, and Section 6 to the technique of cycle pointing that is formulated using these notions. Section 7 recalls the concept of (Pólya-)Boltzmann samplers, which builds a bridge from combinatorial structures to random algorithms that sample these structures. An emphasis is put on addressing a broad audience by properly introducing all combinatorial tools. Section 8 presents the cycle pointing decomposition of trees and uses this decomposition to collect basic enumerative facts. In Section 9 we apply these combinatorial tools in order to prove our main results.

2. DISCRETE TREES

2.1. Graph theoretic notions. A (labelled) graph G consists of a non-empty set $V(G)$ of *vertices* (or *labels*) and a set $E(G)$ of *edges* that are two-element subsets of $V(G)$. The cardinality $|V(G)|$ of the vertex set is termed the *size* of G . Two vertices $v, w \in V(G)$ are said to be *adjacent* if $\{v, w\} \in E(G)$. An edge $e \in E(G)$ is adjacent to v if $v \in e$. The cardinality of the set of all edges adjacent to a vertex v is termed its *degree* and denoted by $d_G(v)$. A *path* P is a graph such that

$$V(P) = \{v_0, \dots, v_\ell\}, \quad E(P) = \{v_0v_1, \dots, v_{\ell-1}v_\ell\}$$

with the v_i being distinct. The number of edges of a path is its *length*. We say P *connects* or *joins* its endvertices v_0 and v_ℓ and we often write $P = v_0v_1 \dots v_\ell$. If P has length at least two we call the graph $C_\ell = P + v_0v_\ell$ obtained by adding the edge v_0v_ℓ a *cycle*. We say the graph G is *connected* if any two vertices $u, v \in V(G)$ are connected by a path in G . The length of a shortest path connecting the vertices u and v is called the *graph distance* of u and v and it is denoted by $d_G(u, v)$. Clearly d_G is a metric on the vertex set $V(G)$. A graph G together with a distinguished vertex $v \in V(G)$ is called a *rooted* graph with root-vertex v . The *height* $h(w)$ of a vertex $w \in V(G)$ is its distance from the root. The *height* $H(G)$ of the entire graph is the supremum of the heights of the vertices in G . Two graphs G_1 and G_2

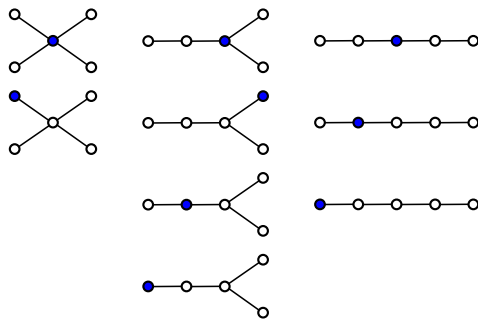


Figure 2. All Pólya trees with 5 vertices.

are termed *isomorphic*, if there is a bijection $\varphi : V(G_1) \rightarrow V(G_2)$ such that any two vertices $x, y \in V(G_1)$ are adjacent in G_1 if and only if $\phi(x)$ and $\phi(y)$ are adjacent in G_2 . Any such bijection is termed an *isomorphism* between G_1 and G_2 . Rooted graphs $G_1^\bullet = (G_1, o_1)$ and $G_2^\bullet = (G_2, o_2)$ are termed isomorphic, if there is a graph isomorphism ϕ from G_1 to G_2 that satisfies $\phi(o_1) = o_2$. An isomorphism class of (rooted) graphs is also called an *unlabelled (rooted) graph*. We will often not distinguish between such a class or any fixed representative of that class.

A *tree* T is a non-empty connected graph without cycles. Any two vertices of a tree are connected by a unique path. Figure 1 depicts the list of all unlabelled trees with 6 vertices. If T is rooted, then the vertices $w' \in V(T)$ that are adjacent to a vertex w and have height $h(w') = h(w) + 1$ form the *offspring set* of the vertex w . Its cardinality is the *outdegree* $d^+(w)$ of the vertex w . Unlabelled rooted trees are also termed Pólya trees. Note that while any labelled tree with n vertices admits n different roots, this does not hold in the unlabelled setting. For example, as illustrated in Figure 2, there are 3 unlabelled trees with 5 vertices and each of them has a different number of rootings.

3. SCALING LIMITS OF RANDOM ROOTED TREES

We briefly recall several relevant results regarding the convergence of random rooted trees toward the continuum random tree.

3.1. Gromov-Hausdorff convergence. We introduce the required notions regarding the Gromov-Hausdorff convergence following Burago, Burago and Ivanov [BBI01, Ch. 7] and Le Gall and Miermont [LGM12]

3.1.1. The Hausdorff metric. Recall that given subsets A and B of a metric space (X, d) , their *Hausdorff-distance* is given by

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subset U_\epsilon(B), B \subset U_\epsilon(A)\} \in [0, \infty],$$

where $U_\epsilon(A) = \{x \in X \mid d(x, A) \leq \epsilon\}$ denotes the ϵ -*hull* of A . In general, the Hausdorff-distance does not define a metric on the set of all subsets of X , but it does on the set of all compact subsets of X ([BBI01, Prop. 7.3.3]).

3.1.2. The Gromov-Hausdorff distance. The Gromov-Hausdorff distance allows us to compare arbitrary metric spaces, instead of only subsets of a common metric space. It is defined by the infimum of Hausdorff-distances of isometric copies in a common metric space. We are also

going to consider a variation of the Gromov-Hausdorff distance given in [LGM12] for *pointed* metric spaces, which are metric spaces together with a distinguished point.

Given metric spaces (X, d_X) , and (Y, d_Y) , and distinguished elements $x_0 \in X$ and $y_0 \in Y$, the Gromov-Hausdorff distances of X and Y and the pointed spaces $X^\bullet = (X, x_0)$ and $Y^\bullet = (Y, y_0)$ are defined by

$$d_{GH}(X, Y) = \inf_{\iota_X, \iota_Y} d_H(\iota_X(X), \iota_Y(Y)) \in [0, \infty],$$

$$d_{GH}(X^\bullet, Y^\bullet) = \inf_{\iota_X, \iota_Y} \max \{d_H(\iota_X(X), \iota_Y(Y)), d_E(\iota_X(x_0), \iota_Y(y_0))\} \in [0, \infty]$$

where in both cases the infimum is taken over all isometric embeddings $\iota_X : X \rightarrow E$ and $\iota_Y : Y \rightarrow E$ into a common metric space (E, d_E) , compare with Figure 3.

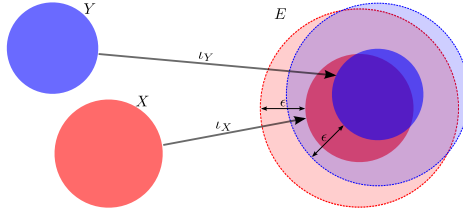


Figure 3. The Gromov-Hausdorff distance.

We will make use of the following characterisation of the Gromov-Hausdorff metric. Given two metric spaces (X, d_X) and (Y, d_Y) a *correspondence* between them is a relation $R \subset X \times Y$ such that any point $x \in X$ corresponds to at least one point $y \in Y$ and vice versa. If X and Y are pointed, we additionally require that the roots correspond to each other. The *distortion* of R is given by

$$\text{dis}(R) = \sup \{|d_X(x_1, x_2) - d_Y(y_1, y_2)| \mid (x_1, y_1), (x_2, y_2) \in R\}.$$

Proposition 3.1 ([BBI01, Thm. 7.3.25] and [LGM12, Prop. 3.6]). *Given two metric spaces X, Y and pointed metric spaces X^\bullet, Y^\bullet we have that*

$$d_{GH}(X, Y) = \frac{1}{2} \inf_R \text{dis}(R), \quad \text{and} \quad d_{GH}(X^\bullet, Y^\bullet) = \frac{1}{2} \inf_R \text{dis}(R),$$

where R ranges over all correspondences between X and Y (or X^\bullet and Y^\bullet).

Using this reformulation of the Gromov-Hausdorff distance, one may check that it satisfies the following properties.

Lemma 3.2 ([BBI01, Thm. 7.3.30] and [LGM12, Thm. 3.5]). *Let X, Y , and Z be (pointed) metric spaces. Then the following assertions hold.*

- i) $d_{GH}(X, Y) = 0$ if and only if X and Y are isometric.
- ii) $d_{GH}(X, Z) \leq d_{GH}(X, Y) + d_{GH}(Y, Z)$.
- iii) If X and Y are bounded, then $d_{GH}(X, Y) < \infty$.

3.1.3. The space of isometry classes of compact metric spaces. In Section 3.1.1 we saw that the Hausdorff-distance defines a metric on the set of all compact subsets of a metric space. By Lemma 3.2 the Gromov-Hausdorff distance satisfies in a similar way the axioms of a (finite) pseudo-metric on the class of all compact metric spaces, and two metric spaces have Gromov-Hausdorff distance 0 if and only if they are isometric. Informally speaking, this yields a metric

on the collection of all isometry classes of metric spaces, and in a similar way we may endow the collection of isometry classes of pointed metric spaces with a metric.

Note that from a formal viewpoint this construction is a bit problematic, since we are forming a collection of proper classes (as opposed to sets). A solution is presented as an exercise in [BBI01, Rem. 7.2.5]:

Proposition 3.3. *Any set of pairwise non-isometric (pointed) metric spaces has cardinality at most 2^{\aleph_0} , and there are specific examples of 2^{\aleph_0} many non-isometric (pointed) spaces.*

We may thus fix a representative of each isometry class of (pointed) metric spaces and let \mathbb{K} (resp. \mathbb{K}^\bullet) denote the resulting sets of spaces. Lemma 3.2 now reads as follows.

Corollary 3.4 ([BBI01, Thm. 7.3.30]). *The Gromov-Hausdorff distance defines a finite metric on the set \mathbb{K} (resp. \mathbb{K}^\bullet) of representatives of isometry classes of (pointed) compact metric spaces.*

The metric spaces \mathbb{K} and \mathbb{K}^\bullet have nice properties, which make them very suitable for studying random elements:

Proposition 3.5 ([LGM12, Thm. 3.5] and [BBI01, Thm. 7.4.15]). *The spaces \mathbb{K} and \mathbb{K}^\bullet are separable and complete, i.e. they are Polish spaces.*

3.2. The continuum random tree. An \mathbb{R} -tree is a metric space (X, d) such that for any two points $x, y \in X$ the following properties hold

1. There is a unique isometric map from the interval $\varphi_{x,y} : [0, d_f(x, y)] \rightarrow X$ satisfying $\varphi_{x,y}(0) = x$ and $\varphi_{x,y}(d_f(x, y)) = y$.
2. If $q : [0, d_f(x, y)] \rightarrow X$ is a continuous injective map, then

$$q([0, d_f(x, y)]) = \varphi_{x,y}([0, d_f(x, y)]).$$

\mathbb{R} -trees may be constructed as follows. Let $f : [0, 1] \rightarrow [0, \infty[$ be a continuous function satisfying $f(0) = f(1) = 0$. Consider the pseudo-metric d on the interval $[0, 1]$ given by

$$d(u, v) = f(u) + f(v) - 2 \inf_{u \leq s \leq v} f(s)$$

for $u \leq v$. Let $(\mathcal{T}_f, d_{\mathcal{T}_f}) = ([0, 1]/\sim, \bar{d})$ denote the corresponding quotient space. We may consider this space as rooted at the equivalence class $\bar{0}$ of 0.

Proposition 3.6 ([LGM12, Thm. 3.1]). *Given a continuous function $f : [0, 1] \rightarrow [0, \infty[$ satisfying $f(0) = f(1) = 0$ the corresponding metric space \mathcal{T}_f is a compact \mathbb{R} -tree.*

Hence, this construction defines a map from a set of continuous functions to the space \mathbb{K}^\bullet . It can be seen to be Lipschitz-continuous:

Proposition 3.7 ([LGM12, Cor. 3.7]). *The map*

$$(\{f \in \mathcal{C}([0, 1], \mathbb{R}_{\geq 0}) \mid f(0) = f(1) = 0\}, \|\cdot\|_\infty) \rightarrow (\mathbb{K}^\bullet, d_{GH}), \quad f \mapsto \mathcal{T}_f$$

is Lipschitz-continuous.

Hence we may define the continuum random tree as a random element of the polish space \mathbb{K}^\bullet .

Definition 3.8. *The random pointed metric space $(\mathcal{T}_e, d_{\mathcal{T}_e}, \bar{0})$ coded by the Brownian excursion of duration one $e = (e_t)_{0 \leq t \leq 1}$ is called the Brownian continuum random tree (CRT).*

Note that the Lipschitz-continuity (and hence measurability) of the above map ensures that the CRT is a random variable.

3.3. Convergence of random rooted trees. Aldous' classical result [Ald91b, Ald91c, Ald93] treats the convergence of random labelled trees and Galton-Watson trees, establishing

$$(\mathcal{T}_n, \frac{\sigma}{2} n^{-1/2} d_{\mathcal{T}_n}) \xrightarrow{(d)} (\mathcal{T}_e, d_{\mathcal{T}_e})$$

as n becomes large, with \mathcal{T}_n denoting a Galton-Watson tree conditioned on having n vertices, with the offspring distribution being critical and having finite nonzero variance $0 < \sigma^2 < \infty$. (To be precise, Aldous' made the additional requirement that ξ has to be aperiodic, but this requirement may safely be dropped [LG10, Thm 6.1].)

It is also known that for any subset $\Omega^* \subset \mathbb{N}_0$ containing zero and at least one integer $k \geq 2$, the Pólya tree A_n drawn uniformly at random from the set of all Pólya trees with n vertices and vertex outdegrees in the set Ω^* admits the CRT as scaling limit. That is, there is a constant c_{Ω^*} satisfying

$$(3.1) \quad (A_n, c_{\Omega^*} n^{-1/2} d_{A_n}) \xrightarrow{(d)} (\mathcal{T}_e, d_{\mathcal{T}_e}).$$

This has been shown by Marckert and Miermont for the case $\Omega^* = \{0, 2\}$ in [MM11]. Using different techniques that built on general results for Markov branching trees, Haas and Miermont [HM12] extend this result to the cases $\Omega^* = \{0, d\}$ for all $d \geq 2$, and $\Omega^* = \mathbb{N}_0$. A unified approach for all sensible vertex outdegree restrictions (that is, requiring only $0 \in \Omega^*$ and $k \in \Omega^*$ for at least one $k \geq 2$) was taken in [PS15], using combinatorial techniques and obtaining tailbounds for the diameter $D(A_n)$ of the form

$$(3.2) \quad \mathbb{P}(D(A_n) \geq x) \leq C \exp(-cx^2)$$

for all $x \geq 0$.

4. LOCAL WEAK LIMITS OF RANDOM ROOTED TREES

We briefly recall relevant notions and results regarding the local convergence of random rooted trees.

4.1. The metric for local convergence. Given two rooted, locally finite (that is, the graph may have infinitely many vertices, but each vertex has only finitely many neighbours) connected graphs $G^\bullet = (G, o_G)$ and $H^\bullet = (H, o_H)$, we may consider the distance

$$d_{BS}(G^\bullet, H^\bullet) = 2^{-\sup\{k \in \mathbb{N}_0 \mid V_k(G^\bullet) \simeq V_k(H^\bullet)\}}$$

with $V_k(G^\bullet)$ denoting the subgraph of G induced by all vertices with graph-distance at most k from the root-vertex o_G . Here $V_k(G^\bullet) \simeq V_k(H^\bullet)$ denotes isomorphism of rooted graphs, that is, the existence of a graph isomorphism $\phi : V_k(G^\bullet) \rightarrow V_k(H^\bullet)$ satisfying $\phi(o_G) = o_H$. This defines a premetric on the collection of all rooted locally finite graphs.

If \mathbb{B} denotes the collection of isomorphism classes of rooted locally finite connected graphs ("unlabelled rooted graphs"), then the (lift of) this distance defines a metric on \mathbb{B} which is complete and separable, i.e. (\mathbb{B}, d_{BS}) is a Polish space. Similarly as for the Gromov-Hausdorff metric, we may safely ignore the fact that \mathbb{B} is a collection of proper classes (as opposed to sets). In order to precise, we would only need to fix a representatives of each isomorphism class and work with the set of these representatives instead.

4.2. Convergence of random rooted trees. Let \mathcal{T}_n denote a ξ -Galton-Watson tree conditioned on having n vertices, such that the offspring distribution ξ satisfies $\mathbb{E}[\xi] = 1$ and $0 < \mathbb{V}[\xi] < \infty$. The Kesten tree $\hat{\mathcal{T}}$ is the classical limit object of Galton-Watson trees conditioned to be large. With respect to the fixed roots of \mathcal{T}_n and $\hat{\mathcal{T}}$, it holds that

$$(\mathcal{T}_n, \text{root}) \xrightarrow{(d)} (\hat{\mathcal{T}}, \text{root}).$$

This is implicit in Kennedy [Ken75] (who considered Galton-Watson processes, rather than trees) and explicit in Aldous and Pitman [AP98]. It's construction is as follows. Consider the *size-biased* random variable $\hat{\xi}$ with distribution given by

$$\mathbb{P}(\hat{\xi} = k) = k\mathbb{P}(\xi = k).$$

There are two types of vertices, normal and mutant. Normal vertices have offspring according to an independent copy of ξ , and each of these is normal again. Mutant vertices have offspring according to an independent copy of $\hat{\xi}$, and each of these offspring vertices is normal, except for a uniformly at random drawn exception. The tree starts with a root that is declared mutant.

The weak limit with respect to a uniformly at random drawn root u_n of \mathcal{T}_n is similar, but a priori different. Aldous [Ald91a] showed that

$$(\mathcal{T}_n, u_n) \xrightarrow{(d)} (\hat{\mathcal{T}}^*, \text{root}),$$

with $\hat{\mathcal{T}}^*$ being identically distributed as $\hat{\mathcal{T}}$, except for the vertex degree of the root, which is drawn according to $\xi^* \stackrel{(d)}{=} \xi + 1$ instead of $\hat{\xi}$. A priori, this is a different tree, but there exist examples that are important in this context, for which $\xi^* \stackrel{(d)}{=} \hat{\xi}$ and hence $\hat{\mathcal{T}}^* \stackrel{(d)}{=} \hat{\mathcal{T}}$. For example, for if ξ is Poisson distributed with parameter one. In this case, the tree \mathcal{T}_n corresponds to a random labelled tree with n vertices.

Let $\Omega \subset \mathbb{N}$ denote a subset containing 1 and at least one integer $k \geq 3$, and let $\Omega^* = \Omega - 1$ denote the shifted set. Let \mathbf{A}_n denote the random tree drawn uniformly at random from the set of all Pólya trees with n vertices and vertex outdegrees in Ω^* . Let o_n denote the root of \mathbf{A}_n , and u_n denote a uniformly at random drawn second root. It was shown in [Stu15] by using results of Aldous [Ald91a], that there are random rooted trees (\mathbf{A}_Ω, u) and (\mathbf{B}_Ω, o) , such that

$$(4.1) \quad (\mathbf{A}_n, u_n) \xrightarrow{(d)} (\mathbf{A}_\Omega, u), \quad \text{and} \quad (\mathbf{A}_n, o_n) \xrightarrow{(d)} (\mathbf{B}_\Omega, o).$$

The limit tree \mathbf{A}_Ω may be described as follows. Start with the tree $\hat{\mathcal{T}}^*$ corresponding to a certain offspring distribution ξ that depends on the outdegree restriction set Ω^* . For example, if $\Omega^* = \mathbb{N}_0$, then ξ needs to be Poisson distributed with parameter one, and hence $\hat{\mathcal{T}}^* \stackrel{(d)}{=} \hat{\mathcal{T}}$. If $\Omega^* = \{0, 2\}$, then $\mathbb{P}(\xi = 0) = 1/2 = \mathbb{P}(\xi = 2)$, and hence $\hat{\mathcal{T}}^*$ and $\hat{\mathcal{T}}$ are not identically distributed.

Then, conditionally independent on $\hat{\mathcal{T}}^*$, each vertex $v \in \hat{\mathcal{T}}^*$ is identified with the root of a random tree \mathbf{T}_v . If we do not impose vertex degree restrictions, then those attached trees are identically distributed. But if we do, the distribution of \mathbf{T}_v depends on the outdegree of the parent of v , in order to respect the vertex degree restrictions. Finally, the root vertex u is drawn uniformly at random from the tree \mathbf{T}_{root} corresponding to the root of $\hat{\mathcal{T}}^*$.

The construction of \mathbf{B}_Ω is the same, except that we start with $\hat{\mathcal{T}}$ instead of $\hat{\mathcal{T}}^*$ and let o denote the fixed root of $\hat{\mathcal{T}}$. For the case $\Omega^* = \mathbb{N}_0$, the trees $\mathbf{A}_{\mathbb{N}_0} \stackrel{(d)}{=} \mathbf{B}_{\mathbb{N}_0}$ are identically distributed, but u is drawn at random near the fixed root o .

5. COMBINATORIAL SPECIES OF STRUCTURES

Combinatorial species were developed by Joyal [Joy81] and allow for a systematic study of a wide range of combinatorial objects. We are going to make heavy use of this framework and recall the required theory and notation following Bergeron, Labelle and Leroux [BLL98] and Joyal [Joy81]. The language of *combinatorial classes* used in the book on analytic combinatorics by Flajolet and Sedgewick [FS09] is essentially equivalent in many aspects, although less emphasis is put on studying objects up to symmetry.

5.1. Combinatorial species of structures. A *combinatorial species* may be defined as a functor \mathcal{F} that maps any finite set U of *labels* to a finite set $\mathcal{F}[U]$ of \mathcal{F} -*objects* and any bijection $\sigma : U \rightarrow V$ of finite sets to its (bijective) *transport function* $\mathcal{F}[\sigma] : \mathcal{F}[U] \rightarrow \mathcal{F}[V]$ *along* σ , such that composition of maps and the identity maps are preserved. Formally, a species is a functor from the groupoid of finite sets and bijections to the category of finite sets and arbitrary maps. We say that a species \mathcal{G} is a *subspecies* of \mathcal{F} , and write $\mathcal{G} \subset \mathcal{F}$, if $\mathcal{G}[U] \subset \mathcal{F}[U]$ for all finite sets U and $\mathcal{G}[\sigma] = \mathcal{F}[\sigma]|_{\mathcal{G}[U]}$ for all bijections $\sigma : U \rightarrow V$. Given two species \mathcal{F} and \mathcal{G} , an *isomorphism* $\alpha : \mathcal{F} \xrightarrow{\sim} \mathcal{G}$ from \mathcal{F} to \mathcal{G} is a family of bijections $\alpha = (\alpha_U : \mathcal{F}[U] \rightarrow \mathcal{G}[U])_U$ where U ranges over all finite sets, such that for all bijective maps $\sigma : U \rightarrow V$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}[U] & \xrightarrow{\mathcal{F}[\sigma]} & \mathcal{F}[V] \\ \downarrow \alpha_U & & \downarrow \alpha_V \\ \mathcal{G}[U] & \xrightarrow{\mathcal{G}[\sigma]} & \mathcal{G}[V] \end{array}$$

In other words, α is a natural isomorphism between these functors. The species \mathcal{F} and \mathcal{G} are *isomorphic* if there exists an isomorphism from one to the other. This is denoted by $\mathcal{F} \simeq \mathcal{G}$ or, by an abuse of notation committed frequently in the literature, just $\mathcal{F} = \mathcal{G}$. Formally, we may form the groupoid of combinatorial species with its objects given by species and its morphisms by natural isomorphisms.

An element $F_U \in \mathcal{F}[U]$ has size $|F_U| := |U|$ and two \mathcal{F} -objects F_U and F_V are termed *isomorphic* if there is a bijection $\sigma : U \rightarrow V$ such that $\mathcal{F}[\sigma](F_U) = F_V$. We will often just write $\sigma.F_U = F_V$ instead, if there is no risk of confusion. We say σ is an *isomorphism* from F_U to F_V . If $U = V$ and $F_U = F_V$ then σ is an *automorphism* of F_U . An isomorphism class of \mathcal{F} -structures is called an *unlabelled \mathcal{F} -object* or an *isomorphism type*.

5.1.1. Examples. We will mostly be interested in the species of labelled trees. Moreover, we will make use of standard species such as the SET-species given by $\text{SET}[U] = \{U\}$ for all U . Moreover, we let \mathcal{X} the species with a single object of size 1.

5.2. Symmetries and generating power series. Letting \tilde{f}_n denote the number of unlabelled \mathcal{F} -objects of size n , the *ordinary generating series* of \mathcal{F} is defined by

$$\tilde{\mathcal{F}}(x) = \sum_{n=0}^{\infty} \tilde{f}_n x^n$$

A pair (F, σ) of an \mathcal{F} -object together with an automorphism is called a *symmetry*. Its *weight monomial* is given by

$$w_{(F, \sigma)} = \frac{1}{n!} x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_n^{\sigma_n} \in \mathbb{Q}[[x_1, x_2, \dots]]$$

with n denoting the size of F and σ_i denoting the number of i -cycles of the permutation σ . In particular σ_1 denotes the number of fixpoints. We may form the species $\text{Sym}(\mathcal{F})$ of symmetries of \mathcal{F} . The *cycle index sum* of \mathcal{F} is given by

$$Z_{\mathcal{F}} = \sum_{(F, \sigma)} w_{(F, \sigma)}$$

with the sum index (F, σ) ranging over the set $\bigcup_{n \in \mathbb{N}_0} \text{Sym}(\mathcal{F})[n]$. The reason for studying cycle index sums is the following remarkable property.

Lemma 5.1 ([Joy81, Sec. 3]). *Let U be a finite n -element set. For any unlabelled \mathcal{F} -object m of size n there are precisely $n!$ symmetries $(F, \sigma) \in \text{Sym}(\mathcal{F})[U]$ having the property that F has isomorphism type m .*

The prove of Lemma 5.1 is an application of basic facts of operations of groups on finite sets. From a probabilistic viewpoint, it guarantees that the isomorphism type of the first coordinate of a uniformly at random drawn element from $\text{Sym}(\mathcal{F})[n]$ is uniformly distributed among all n -element unlabelled \mathcal{F} -objects. This is crucial, as symmetries may be decomposed fairly systematically using the theory of species.

Lemma 5.1 implies that the ordinary generating series and the cycle index sum are related by

$$\tilde{\mathcal{F}}(z) = Z_{\mathcal{F}}(z, z^2, z^3, \dots).$$

See also [Joy81, Sec. 3, Prop. 9].

5.2.1. Examples. The cycle index sum Z_{SET} is easily calculated: For any integer $n \geq 0$ let \mathcal{S}_n denote the symmetric group of order n . Then

$$Z_{\text{SET}} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_n^{\sigma_n}.$$

For any permutation σ let $(\sigma_1, \sigma_2, \dots) \in (\mathbb{N}_0)^{\mathbb{N}}$ denote its *cycle type*. Then to each element $m = (m_i)_i \in \mathbb{N}_0^{\mathbb{N}}$ correspond only permutations of order $n := \sum_{i=1}^{\infty} i m_i$ and their number is given by $n! / \prod_{i=1}^{\infty} (m_i! i^{m_i})$. Hence we have

$$Z_{\text{SET}} = \sum_{m \in \mathbb{N}_0^{\mathbb{N}}} \prod_{i=1}^{\infty} \frac{x_i^{m_i}}{m_i! i^{m_i}} = \prod_{i=1}^{\infty} \sum_{m_i=0}^{\infty} \frac{x_i^{m_i}}{m_i! i^{m_i}} = \prod_{i=1}^{\infty} \exp\left(\frac{x_i}{i}\right) = \exp\left(\sum_{i=1}^{\infty} \frac{x_i}{i}\right).$$

If $(x_i)_i$ would denote a sequence of sufficiently fast decaying positive real-numbers, then this calculation could easily be justified. But they denote a countable set of formal variables, and hence one has every right to ask for a rigorous justification of this argument, in particular why the involved infinite products of formal variables vanish. We refer the inclined reader to [FS09, Appendix A.5] for an adequate discussion of these questions.

5.3. Operations on combinatorial species. The framework of combinatorial species offers a large variety of constructions that create new species from others. In the following let \mathcal{F} , $(\mathcal{F}_i)_{i \in \mathbb{N}}$ and \mathcal{G} denote species and U an arbitrary finite set. The *sum* $\mathcal{F} + \mathcal{G}$ is defined by the disjoint union

$$(\mathcal{F} + \mathcal{G})[U] = \mathcal{F}[U] \sqcup \mathcal{G}[U].$$

More generally, the infinite sum $(\sum_i \mathcal{F}_i)$ may be defined by $(\sum_i \mathcal{F}_i)[U] = \bigsqcup_i \mathcal{F}_i[U]$ if the right hand side is finite for all finite sets U . The *product* $\mathcal{F} \cdot \mathcal{G}$ is defined by the disjoint union

$$(\mathcal{F} \cdot \mathcal{G})[U] = \bigsqcup_{\substack{(U_1, U_2) \\ U_1 \cap U_2 = \emptyset, U_1 \cup U_2 = U}} \mathcal{F}[U_1] \times \mathcal{G}[U_2]$$

with componentwise transport. Thus, n -sized objects of the product are pairs of \mathcal{F} -objects and \mathcal{G} -objects whose sizes add up to n . If the species \mathcal{G} has no objects of size zero, we can form the *substitution* $\mathcal{F} \circ \mathcal{G}$ by

$$(\mathcal{F} \circ \mathcal{G})[U] = \bigsqcup_{\pi \text{ partition of } U} \mathcal{F}[\pi] \times \prod_{Q \in \pi} \mathcal{G}[Q].$$

An object of the substitution may be interpreted as an \mathcal{F} -object whose labels are substituted by \mathcal{G} -objects. The transport along a bijection σ is defined by applying the induced map $\bar{\sigma} : \pi \rightarrow \bar{\pi} = \{\sigma(Q) \mid Q \in \pi\}$ of partitions to the \mathcal{F} -object and the restricted maps $\sigma|_Q$ with $Q \in \pi$ to their corresponding \mathcal{G} -objects. We will often write $\mathcal{F}(\mathcal{G})$ instead of $\mathcal{F} \circ \mathcal{G}$. Explicit formulas for the generating series and cycle index sums of the discussed constructions are summarized in Table 1. The notation is quite suggestive: up to (canonical) isomorphism, each operation considered in this section is associative. Roughly described, this means that for each operation $\mu \in \{+, \cdot, \circ\}$ there is a "natural choice" for an isomorphism

$$(\mathcal{F}_1 \mu \mathcal{F}_2) \mu \mathcal{F}_3 \simeq \mathcal{F}_1 \mu (\mathcal{F}_2 \mu \mathcal{F}_3).$$

But this is only half of the story: for example, we may apply these isomorphisms in different orders in order to obtain an isomorphism from $((\mathcal{F}_1 \mu \mathcal{F}_2) \mu \mathcal{F}_3) \mu \mathcal{F}_4$ to $\mathcal{F}_1 \mu (\mathcal{F}_2 \mu (\mathcal{F}_3 \mu \mathcal{F}_4))$. But why should we end up with the same isomorphism, regardless of which order we choose? In order to answer this question adequately, the concept of monoidal categories is required, and we refer the inclined reader to [Joy81, Sec. 7] for a thorough discussion.

The sum and product are commutative operations (up to canonical isomorphisms) and satisfy the distributive law

$$(5.1) \quad \mathcal{F} \cdot (\mathcal{G}_1 + \mathcal{G}_2) \simeq \mathcal{F} \cdot \mathcal{G}_1 + \mathcal{F} \cdot \mathcal{G}_2.$$

for any two species \mathcal{G}_1 and \mathcal{G}_2 . The operation of deriving a species is additive and satisfies a product rule and a chain rule, analogous to the derivative in calculus:

$$(5.2) \quad (\mathcal{F} \cdot \mathcal{G})' \simeq \mathcal{F}' \cdot \mathcal{G} + \mathcal{F} \cdot \mathcal{G}' \quad \text{and} \quad \mathcal{F}(\mathcal{G})' \simeq \mathcal{F}'(\mathcal{G}) \cdot \mathcal{G}'.$$

Recall that for the chain rule to apply we have to require $\mathcal{G}[\emptyset] = \emptyset$, since otherwise $\mathcal{F}(\mathcal{G})$ is not defined.

5.4. Decomposition of symmetries of the substitution operation. We are going to need detailed information on the structure of the symmetries of the composition $\mathcal{F} \circ \mathcal{G}$. The following is a standard decomposition given in [Joy81, BLL98, BFKV11]. Let U be a finite set. Any element of $\text{Sym}(\mathcal{F} \circ \mathcal{G})[U]$ consists of the following objects: a partition π of the set U , a \mathcal{F} -structure $F \in \mathcal{F}[\pi]$, a family of \mathcal{G} -structures $(G_Q)_{Q \in \pi}$ with $G_Q \in \mathcal{G}[Q]$ and a permutation $\sigma : U \rightarrow U$. We require the permutation σ to permute the partition classes and induce an

	OGF	Cycle index sum
$\sum_i \mathcal{F}_i$	$\sum_i \tilde{\mathcal{F}}_i(x)$	$\sum_i Z_{\mathcal{F}_i}(x_1, x_2, \dots)$
$\mathcal{F} \cdot \mathcal{G}$	$\tilde{\mathcal{F}}(x) \tilde{\mathcal{G}}(x)$	$Z_{\mathcal{F}}(x_1, x_2, \dots) Z_{\mathcal{G}}(x_1, x_2, \dots)$
$\mathcal{F} \circ \mathcal{G}$	$Z_{\mathcal{F}}(\tilde{\mathcal{G}}(x), \tilde{\mathcal{G}}(x^2), \dots)$	$Z_{\mathcal{F}}(Z_{\mathcal{G}}(x_1, x_2, \dots), Z_{\mathcal{G}}(x_2, x_4, \dots), \dots)$

Table 1. Relation between combinatorial constructions and generating series.

automorphism $\bar{\sigma} : \pi \rightarrow \pi$ of the \mathcal{F} -object F . Moreover, for any partition class $Q \in \pi$ we require that the restriction $\sigma|_Q : Q \rightarrow \sigma(Q)$ is an isomorphism from G_Q to $G_{\sigma(Q)}$. For any cycle $\bar{\tau} = (Q_1, \dots, Q_\ell)$ of $\bar{\sigma}$ it follows that for all i we have $\sigma^\ell(Q_i) = Q_i$ and the restriction $\sigma^\ell|_{Q_i} : Q_i \rightarrow Q_i$ is an automorphism of G_{Q_i} . Conversely, if we know $(G_{Q_1}, \sigma^\ell|_{Q_1})$ and the maps $\sigma|_{Q_i} = (\sigma|_{Q_1})^i$ for $1 \leq i \leq \ell - 1$, we can reconstruct the \mathcal{G} -objects $G_{Q_2}, \dots, G_{Q_\ell}$ and the restriction $\sigma|_{Q_1 \cup \dots \cup Q_\ell}$. Here any k -cycle (a_1, \dots, a_k) of the permutation $\sigma^\ell|_{Q_1}$ corresponds to the $k\ell$ -cycle

$$(a_1, \sigma(a_1), \dots, \sigma^{\ell-1}(a_1), a_2, \sigma(a_2), \dots, \sigma^{\ell-1}(a_2), \dots, a_k, \sigma(a_k), \dots, \sigma^{\ell-1}(a_k))$$

of $\sigma|_{Q_1 \cup \dots \cup Q_\ell}$. Thus any cycle ν of σ corresponds to a cycle of the induced permutation $\bar{\sigma}$ whose length is a divisor of the length of ν .

6. CYCLE POINTING

Cycle pointing is a technique introduced by Bodirsky, Fusy, Kang and Vigerske [BFKV11] as a means to study unlabelled graphs and trees. One of their main application is to the enumeration of unlabelled unrooted trees, providing a new proof for their asymptotic enumeration formula, that does not require the dissymmetry theorem.

6.1. The cycle pointing operator. Bodirsky, Fusy, Kang and Vigerske [BFKV11] introduced the cycle pointing operator which maps a species \mathcal{G} to the species \mathcal{G}° such that the \mathcal{G}° -objects over a set U are pairs (G, τ) with $G \in \mathcal{G}[U]$ and τ a *marked* cycle of an arbitrary automorphism of G . Here we count fixpoints as 1-cycles. The transport is defined by $\sigma.(G, \tau) = (\sigma.G, \sigma\tau\sigma^{-1})$. Any subspecies $\mathcal{S} \subset \mathcal{G}^\circ$ is termed *cycle-pointed*. The *symmetric* cycle-pointed species $\mathcal{G}^\circ \subset \mathcal{G}^\circ$ is defined by restricting to pairs (G, τ) with τ a cycle of length at least 2.

A *rooted c-symmetry* of the cycle-pointed species $\mathcal{S} \subset \mathcal{G}^\circ$ is a quadruple $((G, \tau), \sigma, v)$ such that (G, τ) is a \mathcal{S} -object, σ is an automorphism of G , τ is a cycle of σ and v is an atom of the cycle τ . Its *weight monomial* is given by

$$w_{((G, \tau), \sigma, v)} = \frac{t_\ell}{s_\ell} w_{(G, \sigma)}(s_1, s_2, \dots)$$

with $w_{(G, \sigma)}$ denoting the weight of the symmetry (G, σ) and ℓ the length of the marked cycle τ . We may form the species $\text{RSym}(\mathcal{S})$ of rooted c -symmetries of \mathcal{S} . The pointed cycle index sum of \mathcal{S} is given by

$$\bar{Z}_{\mathcal{S}}(s_1, t_1; s_2, t_2; \dots) = \sum_{(G, \tau, \sigma, v)} w_{(G, \tau, \sigma, v)} \in \mathbb{Q}[[s_1, t_1; s_2, t_2; \dots]]$$

with the index ranging over the set $\bigcup_{n \in \mathbb{N}_0} \text{RSym}(\mathcal{S})[n]$.

Let $\mathcal{G}_{(\ell)}^\circ \subset \mathcal{G}^\circ$ denote the subspecies given by all cycle pointed objects whose marked cycle has length ℓ . It follows from the definition of the pointed cycle index sum that

$$\bar{Z}_{\mathcal{G}_{(\ell)}^\circ} = \ell t_\ell \frac{\partial}{\partial s_\ell} Z_{\mathcal{G}}.$$

Since $\mathcal{G}^\circ = \sum_{\ell=1}^{\infty} \mathcal{G}_{(\ell)}^\circ$ it follows that

$$\bar{Z}_{\mathcal{G}^\circ} = \sum_{\ell=1}^{\infty} \ell t_\ell \frac{\partial}{\partial s_\ell} Z_{\mathcal{G}} \quad \text{and} \quad \bar{Z}_{\mathcal{G}^\circ} = \sum_{\ell=2}^{\infty} \ell t_\ell \frac{\partial}{\partial s_\ell} Z_{\mathcal{G}}.$$

Lemma 6.1 ([BFKV11, Lem. 14]). *Let U be a finite set with n elements and fix an arbitrary linear order on U .*

1) *The following map is bijective:*

$$\begin{aligned} R\text{Sym}(\mathcal{S})[U] &\rightarrow \text{Sym}(\mathcal{S})[U], \\ M = ((G, \tau), \sigma, v) &\mapsto ((\tau^{1-\ell(M)} \cdot G, \tau), \sigma \tau^{\ell(M)-1}) \end{aligned}$$

with $\ell(M)$ defined as follows: let k denote the length of the cycle τ and u its smallest atom. Let $0 \leq \ell(M) \leq k-1$ be the unique integer satisfying $v = \tau^{\ell(M)} \cdot u$.

2) *Any unlabelled cycle-pointed \mathcal{S} -object m of size n corresponds to precisely $n!$ rooted c -symmetries from $R\text{Sym}(\mathcal{S})[U]$ having the property that the isomorphism type of the underlying \mathcal{S} -object equals m .*

In particular, the pointed cycle index sum relates to the ordinary generating series by

$$\tilde{\mathcal{S}}(x) = \bar{Z}_{\mathcal{S}}(x, x; x^2, x^2; \dots).$$

Moreover, if we draw an element from $R\text{Sym}(\mathcal{S})[n]$ uniformly at random, then the isomorphism class of the corresponding cycle pointed structure is uniformly distributed among all unlabelled \mathcal{S} -objects of size n .

The main point of the cycle-pointing construction is evident from the following fact.

Lemma 6.2 ([BFKV11, Thm. 15]). *Any unlabelled \mathcal{G} -structure m of size n may be cycle-pointed in precisely n ways, i.e. there exist precisely n unlabelled \mathcal{G}° -structures with corresponding \mathcal{G} -structure m .*

Considered from a probabilistic viewpoint, this means that if we draw an unlabelled \mathcal{G}° -structure of size n uniformly at random, then the underlying \mathcal{G} -object is also uniformly distributed. And studying the random \mathcal{G}° -object might be easier due to the additional information given by the marked cycle. Moreover, Lemma 6.2 implies that

$$\tilde{\mathcal{G}}^\circ(z) = z \frac{d}{dz} \tilde{\mathcal{G}}(z).$$

6.1.1. *Example.* The pointed cycle index sum of the species SET is given by

$$\bar{Z}_{\text{SET}^\circ} = \sum_{\ell=1}^{\infty} \ell t_\ell \frac{\partial}{\partial s_\ell} Z_{\text{SET}}(s_1, s_2, \dots) = \exp\left(\sum_{i=1}^{\infty} s_i/i\right) \sum_{\ell=1}^{\infty} t_i.$$

6.2. Operations on cycle pointed species. Cycle pointed species come with a set of new operations introduced in [BFKV11]. If $\mathcal{S} \subset \mathcal{G}^\circ$ is a cycle-pointed species and \mathcal{H} a species, then the *pointed product* $\mathcal{S} \star \mathcal{H}$ is the subspecies of $(\mathcal{G} \cdot \mathcal{H})^\circ$ given by all cycle-pointed objects such that the marked cycle consists of atoms of the \mathcal{G} -structure and the \mathcal{G} -structure together with this cycle belongs to \mathcal{S} . The corresponding pointed cycle index sum is given by

$$\bar{Z}_{\mathcal{S} \star \mathcal{H}} = \bar{Z}_{\mathcal{S}} Z_{\mathcal{H}}.$$

The cycle-pointing operator obeys the following product rule

$$(\mathcal{G} \cdot \mathcal{H})^\circ \simeq \mathcal{G}^\circ \star \mathcal{H} + \mathcal{H}^\circ \star \mathcal{G}.$$

If $\mathcal{H}[\emptyset] = \emptyset$ we may form the *pointed substitution* $\mathcal{S} \odot \mathcal{H} \subset (\mathcal{G} \odot \mathcal{H})^\circ$ as follows. Any $(\mathcal{G} \odot \mathcal{H})^\circ$ -structure P has a marked cycle τ of some automorphism σ . By the discussion in Section 5.4, this cycle corresponds to a cycle on the \mathcal{G} -structure of P which does not depend on the choice of σ . Hence the \mathcal{G} -structure of P is cycle-pointed and we say P belongs to $\mathcal{S} \odot \mathcal{H}$ if and only if this cycle pointed \mathcal{G} -structure belongs to \mathcal{S} . The corresponding pointed cycle index sum is given by

$$\begin{aligned} \bar{Z}_{\mathcal{S} \odot \mathcal{H}} &= \bar{Z}_{\mathcal{S}}(Z_{\mathcal{H}}(s_1, s_2, \dots), \bar{Z}_{\mathcal{H}^\circ}(s_1, t_1; s_2, t_2; \dots); \\ &\quad Z_{\mathcal{H}}(s_2, s_4, \dots), \bar{Z}_{\mathcal{H}^\circ}(s_2, t_2; s_4, t_4; \dots); \dots). \end{aligned}$$

7. (PÓLYA-)BOLTZMANN SAMPLERS

Boltzmann samplers were introduced in [DFLS02, DFLS04, FFP07] and generalized to Pólya-Boltzmann samplers in [BFKV11]. They form our main tool in the analysis of random discrete objects and we discuss the required notions and properties following these sources.

7.1. Boltzmann models. For any integer n , let $\tilde{\mathcal{F}}[n]$ denote the set of unlabelled \mathcal{F} -objects with size n . Given a number $x \geq 0$ with $0 < \tilde{\mathcal{F}}(x) < \infty$, the Boltzmann distribution for unlabelled objects is a probability distribution on the set $\bigcup_{n=0}^{\infty} \tilde{\mathcal{F}}[n]$ that assigns the probability weight

$$x^n \tilde{\mathcal{F}}(x)^{-1}$$

for each n to each unlabelled \mathcal{F} -structure of size n . The corresponding Boltzmann sampler is denoted by $\Gamma \tilde{\mathcal{F}}(x)$.

The *Pólya-Boltzmann model* was introduced in [BFKV11]: Suppose that we are given a sequence of real numbers $s_1, s_2, \dots \geq 0$ such that $0 < Z_{\mathcal{F}}(s_1, s_2, \dots) < \infty$. Then we may consider the probability distribution on the set $\bigcup_{n=0}^{\infty} \text{Sym}(\mathcal{F})[n]$ that assigns the probability weight

$$w_{(F, \sigma)} Z_{\mathcal{F}}(s_1, s_2, \dots)^{-1} = \frac{s_1^{\sigma_1} s_2^{\sigma_2} \dots}{n!} Z_{\mathcal{F}}(s_1, s_2, \dots)^{-1}.$$

for each n and symmetry $(F, \sigma) \in \text{Sym}(\mathcal{F})[n]$. Here σ_i denotes the number of i -cycles of the permutation σ . The corresponding *Pólya-Boltzmann sampler* is denoted by $\Gamma Z_{\mathcal{F}}(s_1, s_2, \dots)$.

Lemma 5.1 directly implies the following crucial property, that shows how Boltzmann samplers for labelled and unlabelled objects are special cases of Pólya Boltzmann samplers.

Lemma 7.1 ([BFKV11, Lem. 36]). *Consider a species \mathcal{F} having a Pólya-Boltzmann sampler $\Gamma Z_{\mathcal{F}}(s_1, s_2, \dots)$. Then, for any parameter $x \geq 0$ with $0 < \tilde{\mathcal{F}}(x) < \infty$ a Boltzmann sampler $\Gamma \tilde{\mathcal{F}}(x)$ for unlabelled \mathcal{F} -objects is given by taking the isomorphism type of the \mathcal{F} -structure of the random symmetry*

$$\Gamma Z_{\mathcal{F}}(x, x^2, \dots).$$

A Pólya-Boltzmann model for random cycle pointed species is given by a probability measure on random rooted c -symmetries: Let \mathcal{S} be a cycle-pointed species. Given real nonnegative numbers $(s_i, t_i)_{i \geq 1}$ such that $0 < \bar{Z}_{\mathcal{S}}(s_1, t_1; s_2, t_2; \dots) < \infty$ we may consider the probability measure on the set $\bigcup_{n=0}^{\infty} \text{RSym}(\mathcal{S})[n]$ that assigns probability weight

$$w_{((G, \tau), \sigma, v)} \bar{Z}_{\mathcal{S}}(s_1, t_1; s_2, t_2; \dots)^{-1} = \frac{t_{\ell} s_1^{\sigma_1} \dots s_{\ell-1}^{\sigma_{\ell-1}} s_{\ell}^{\sigma_{\ell}-1} s_{\ell+1}^{\sigma_{\ell+1}} s_{\ell+2}^{\sigma_{\ell+2}} \dots}{n! \bar{Z}_{\mathcal{S}}(s_1, t_1; s_2, t_2; \dots)}$$

for each n to each rooted c -symmetry $((G, \tau), \sigma, v) \in \text{RSym}[n]$. Here ℓ denotes the lengths of the marked cycle τ . The corresponding Pólya-Boltzmann sampler of this model is denoted by $\Gamma \bar{Z}_{\mathcal{S}}(s_1, t_1; s_2, t_2; \dots)$.

7.2. Rules for the construction of Boltzmann samplers. Boltzmann samplers are traditionally denoted using some Pseudo-code notation. We are going to deviate from this tradition, by providing more detailed explanations of each step using words rather than improvised code. By this the author hopes to make the material more accessible to a larger audience. In the following we are always going to suppose that \mathcal{F} is a species and x, x_1, x_2, \dots are nonnegative numbers such that sums $\mathcal{F}(x)$, and $Z_{\mathcal{F}}(x_1, x_2, \dots)$ are positive and finite. If \mathcal{F} is cycle-pointed, then we also assume that $s_1, t_1, s_2, t_2, \dots$ are nonnegative numbers such that $\bar{Z}_{\mathcal{F}}(s_1, t_1; s_2, t_2; \dots)$ is positive and finite.

Suppose that we are given a decomposition

$$\mathcal{F} = \mathcal{A} \mu \mathcal{B}$$

with $\mu \in \{+, \cdot, \circ\}$ one of the discussed operations of sum, product and substitution. In order to construct a (Pólya-)Boltzmann-sampler for the species \mathcal{F} , we may apply certain construction rules in order to obtain a sampler in terms of samplers for the species \mathcal{A} and \mathcal{B} . In the following we summarize these construction rules for (Pólya-)Boltzmann samplers, following [DFLS04], [FFP07] and [BFKV11].

7.2.1. Pólya-Boltzmann samplers.

Sums. Suppose that $\mathcal{F} = \sum_{i=1}^{\infty} \mathcal{F}_i$. Then the following procedure is a Pólya-Boltzmann sampler for \mathcal{F} .

1. Draw an integer $\ell \geq 1$ with probability

$$\mathbb{P}(\ell = i) = Z_{\mathcal{F}_i}(s_1, s_2, \dots) / Z_{\mathcal{F}}(s_1, s_2, \dots).$$

2. Return $\Gamma Z_{\mathcal{F}_\ell}(s_1, s_2, \dots)$. That is, the result is a random \mathcal{F}_ℓ -symmetry following a Pólya-Boltzmann distribution with the parameters (s_1, s_2, \dots) .

Products. Suppose that $\mathcal{F} = \prod_{i=1}^k \mathcal{F}_i$. Then for any finite set U there is a bijection between the set $\text{Sym}(\mathcal{F})[U]$ and tuples (S_1, \dots, S_k) such that S_i is a \mathcal{F}_i -symmetry for all i and the label sets of the S_i partition the set U . This is due to the fact, that given a \mathcal{F} -symmetry $((F_1, \dots, F_k), \sigma) \in \text{Sym}(\mathcal{F})[U]$ the permutation σ must leave the label set Q_i of the \mathcal{F}_i -object F_i invariant and satisfy $\sigma|_{Q_i} \cdot F_i = F_i$, i.e. $(F_i, \sigma|_{Q_i}) \in \text{Sym}(\mathcal{F}_i)[Q_i]$.

The following procedure is a Pólya-Boltzmann sampler for \mathcal{F} .

1. For each $1 \leq i \leq k$ set

$$(F_i, \sigma_i) \leftarrow \Gamma Z_{\mathcal{F}_i}(s_1, s_2, \dots).$$

That is, let $S_i := (F_i, \sigma_i)$, $1 \leq i \leq k$ be independent random variables such that S_i follows a Pólya-Boltzmann distribution for \mathcal{F}_i with parameters s_1, s_2, \dots

2. By the bijection for the symmetries of products, the tuple (S_1, \dots, S_k) corresponds to an \mathcal{F} -symmetry (F, σ) over the (exterior) disjoint union U of the label-sets of the S_i . Make a uniformly at random choice of a bijection ν from U to the set of integers $[n]$ with n denoting the size of U . Return the relabelled symmetry

$$\nu.(F, \sigma) = (\nu.F, \nu\sigma\nu^{-1}).$$

Substitution. Suppose that $\mathcal{F} = \mathcal{G} \circ \mathcal{H}$ with $\mathcal{H}[\emptyset] = \emptyset$. The symmetries of the substitution were discussed in detail in Section 5.4. The following procedure is a Pólya-Boltzmann sampler for \mathcal{F} .

1. Set

$$(G, \sigma) \leftarrow \Gamma Z_{\mathcal{G}}(Z_{\mathcal{H}}(s_1, s_2, \dots), Z_{\mathcal{H}}(s_2, s_4, \dots), \dots).$$

That is, let (G, σ) denote a random \mathcal{G} -symmetry that follows a Pólya-Boltzmann distribution with parameters $Z_{\mathcal{H}}(s_1, s_2, \dots), Z_{\mathcal{H}}(s_2, s_4, \dots), \dots$.

2. For each cycle τ of σ let $|\tau|$ denote its lengths and set

$$(H_{\tau}, \sigma_{\tau}) \leftarrow \Gamma Z_{\mathcal{H}}(s_{|\tau|}, s_{2|\tau|}, \dots).$$

That is, the symmetries $(H_{\tau}, \sigma_{\tau})$, τ cycle of σ , are independent (conditional on σ) and follow Pólya-Boltzmann distributions.

3. For each cycle τ , make $|\tau|$ identical copies of $(H_{\tau}, \sigma_{\tau})$ and assemble a \mathcal{F} -symmetry (F, γ) out of (G, σ) and the copies of the $(H_{\tau}, \sigma_{\tau})$ as described in Section 5.4.
4. Choose bijection ν from the vertex set of (F, γ) to an appropriate sized set of integers $[n]$ and return the relabelled symmetry

$$\nu.(F, \gamma) = (\nu.F, \nu\gamma\nu^{-1}).$$

The Set construction. The following procedure is a Pólya-Boltzmann sampler for $\mathcal{F} = \text{SET}$.

1. Let $(m_i)_{i \in \mathbb{N}}$ be an independent family of integers $m_i \geq 0$ such that m_i follows a Poisson-distribution with parameter s_i/i .
2. The sequence drawn in the previous step belongs almost surely to $\mathbb{N}_0^{\mathbb{N}}$. Let σ be a permutation with cycle type $(m_i)_i$.
3. Make a uniformly at random choice of a bijection ν from the label set of σ to an appropriate sized set of integers $[n]$ and return the SET-symmetry

$$(F, \nu\sigma\nu^{-1})$$

with $F = [n]$ the unique element from $\text{SET}[n] = \{[n]\}$.

7.2.2. Pólya-Boltzmann samplers for cycle-pointed species. In the following, we suppose that \mathcal{F} is a cycle pointed species.

Sums. Suppose that $\mathcal{F} = \sum_{i=1}^{\infty} \mathcal{F}_i$ with cycle-pointed species \mathcal{F}_i . Then the following procedure is a Pólya-Boltzmann sampler for \mathcal{F} .

1. Draw an integer $\ell \geq 1$ with probability

$$\mathbb{P}(\ell = i) = \bar{Z}_{\mathcal{F}_i}(s_1, t_1; s_2, t_2; \dots) / Z_{\mathcal{F}}(s_1, t_1; s_2, t_2; \dots).$$

2. Return $\Gamma \bar{Z}_{\mathcal{F}_\ell}(s_1, t_1; s_2, t_2; \dots)$.

Cycle pointed products. Suppose that $\mathcal{F} = \mathcal{G} \star \mathcal{H}$ with \mathcal{G} a cycle-pointed species and \mathcal{H} a species. Then for any finite set U there is a canonical choice for a bijection between the set $\text{RSym}(\mathcal{F})[U]$ and tuples (S_1, S_2) with S_1 a rooted c -symmetry of \mathcal{G} , S_2 a symmetry of \mathcal{G} , such that the label sets of S_1 and S_2 form a partition of U . The following procedure is a Pólya-Boltzmann sampler for \mathcal{F} .

1. Set

$$S_1 \leftarrow \Gamma \bar{Z}_{\mathcal{G}}(s_1, t_1; s_2, t_2; \dots).$$

2. Set

$$S_2 \leftarrow \Gamma Z_{\mathcal{H}}(s_1, s_2, \dots).$$

3. Let U denote the exterior disjoint union of the label sets of S_1 and S_2 . The tuple (S_1, S_2) corresponds to a rooted c -symmetry S over the set U .
4. Make a uniformly at random choice of a bijection ν from U to the set of integers $[n]$ with n denoting the size of U . Return the relabelled rooted c -symmetry $\nu.S$.

Cycle pointed substitution. Suppose that $\mathcal{F} = \mathcal{G} \odot \mathcal{H}$ with \mathcal{G} cycle-pointed and $\mathcal{H}[\emptyset] = \emptyset$. The symmetries of the substitution were discussed in detail in Section 5.4. The following procedure is a Pólya-Boltzmann sampler for \mathcal{F} .

1. Set

$$((G, \tau_0), \sigma, v) \leftarrow \Gamma \bar{Z}_{\mathcal{G}}(h_1, \bar{h}_1; h_2, \bar{h}_2; \dots)$$

with parameters

$$h_i = Z_{\mathcal{H}}(s_i, s_{2i}, \dots) \quad \text{and} \quad \bar{h}_i = \bar{Z}_{\mathcal{H}^\circ}(s_i, t_i; s_{2i}, t_{2i}; \dots).$$

2. For each unmarked cycle $\tau \neq \tau_0$ of σ let $|\tau|$ denote its lengths and set

$$(H_\tau, \sigma_\tau) \leftarrow \Gamma Z_{\mathcal{H}}(s_{|\tau|}, s_{2|\tau|}, \dots).$$

3. For the marked cycle τ_0 set

$$((H_{\tau_0}, c_{\tau_0}), \sigma_{\tau_0}, v_{\tau_0}) \leftarrow \Gamma Z_{\mathcal{H}^\circ}(s_{|\tau_0|}, t_{|\tau_0|}; s_{2|\tau_0|}, t_{2|\tau_0|}; \dots).$$

4. For each cycle τ of σ (including the marked cycle τ_0), make $|\tau|$ identical copies of (H_τ, σ_τ) , one for each atom of τ . Assemble a \mathcal{F} -symmetry (F, γ) out of (G, σ) and the copies of the (H_τ, σ_τ) as described in Section 5.4. Let c denote the cycle that gets composed out of the $|\tau_0|$ copies of the cycle c_{τ_0} . The marked vertex v_{τ_0} has $|\tau_0|$ copies (one for each atom of τ_0) and we let u denote the copy that corresponds to the marked atom v_0 of τ_0 . Thus

$$((F, c), \gamma, u)$$

is a rooted c -symmetry of \mathcal{F} .

5. Choose bijection ν from the vertex set of $((F, c), \gamma, u)$ to an appropriate sized set of integers $[n]$ and return the relabelled rooted c -symmetry

$$\nu.((F, c), \gamma, u) = ((\nu.F, \nu c \nu^{-1}), \nu \gamma \nu^{-1}, \nu.u).$$

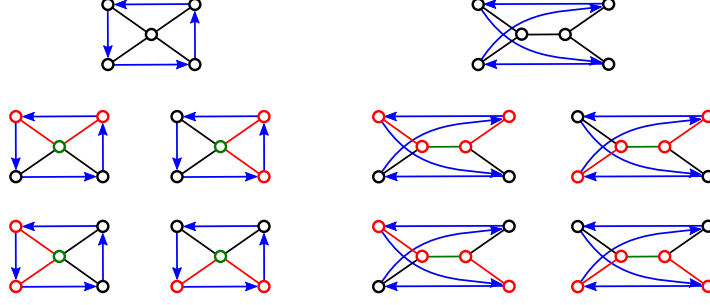


Figure 4. Two unlabelled cycle-pointed trees. The marked cycle is depicted in blue, connecting paths in red, and the cycle-pointing centers in green.

Cycle pointed Set constructions. The following procedure is a Pólya-Boltzmann sampler for $\mathcal{F} = \text{SET}^\circ$.

1. Choose an integer $K \geq 1$ with distribution

$$\mathbb{P}(K = k) = t_k / \sum_{i=1}^{\infty} t_i.$$

2. Set

$$(G, \sigma) \leftarrow \Gamma Z_{\text{SET}}(s_1, s_2, \dots).$$

3. Add a disjoint cycle of length K to the permutation σ . Mark one of the atoms of this cycle uniformly at random.
4. Relabel the resulting rooted c -symmetry uniformly at random.

The sampler for the symmetrically cycle pointed species SET° is identical, only step 1. needs to be replaced with:

- 1'. Choose an integer $K \geq 2$ with distribution

$$\mathbb{P}(K = k) = t_k / \sum_{i=2}^{\infty} t_i.$$

8. CYCLE DECOMPOSITION AND ENUMERATIVE PROPERTIES

Given a cycle pointed tree (T, τ) such that the marked cycle τ has length at least 2 we may consider its *connecting paths*, i.e. the paths in T that join consecutive atoms of τ . Any such path has a middle, which is either a vertex if the path has odd length, or an edge if the path has even length. All connecting paths have the same lengths and by [BFKV11, Claim 22] they share the same middle, called the *center of symmetry*. See Figure 4 for an illustration.

The cycle pointing decomposition given in [BFKV11, Prop. 25] splits the species \mathcal{F}_Ω° into three parts,

$$\mathcal{F}_\Omega^\circ \simeq \mathcal{X}^\circ \star (\text{SET}_\Omega \circ \mathcal{A}_{\Omega^*}) + \text{SET}_{\{2\}}^\circ \odot \mathcal{A}_{\Omega^*} + (\text{SET}_\Omega^\circ \odot \mathcal{A}_{\Omega^*}) \star \mathcal{X}.$$

Here

$$\mathcal{S} := \mathcal{X}^\circ \star (\text{SET}_\Omega \circ \mathcal{A}_{\Omega^*})$$

corresponds to the trees with a marked fixpoint and the other summands to trees with a marked cycle of length at least two. More specifically,

$$\mathcal{E} := \text{SET}_{\{2\}}^\circ \odot \mathcal{A}_{\Omega^*}$$

corresponds to the symmetric cycle pointed trees whose center of symmetry is an edge and

$$\mathcal{V} := (\text{SET}_{\Omega}^{\odot} \odot \mathcal{A}_{\Omega^*}) \star \mathcal{X}$$

to those whose center of symmetry is a vertex.

This decomposition is going to play a key-role in the proofs of our main theorems. We start by collecting some basic enumerative facts.

The following lemma summarizes enumerative properties of Pólya trees with vertex degree restrictions. It is based case on [BBY06, Thm. 75], although not explicitly given this form in [BBY06]. The book by Flajolet and Sedgewick [FS09, Thm. VII.4] covers the aperiodic case.

Proposition 8.1 ([PS15, Prop. 4.1]). *The following statements hold.*

- i) *The radius of convergence ρ of the series $\tilde{\mathcal{A}}_{\Omega^*}(z)$ satisfies $0 < \rho < 1$ and $\tilde{\mathcal{A}}_{\Omega^*}(\rho) < \infty$.*
- ii) *There is a positive constant d_{Ω^*} such that*

$$[z^m] \tilde{\mathcal{A}}_{\Omega^*}(z) \sim d_{\Omega^*} m^{-3/2} \rho^{-m}$$

as the number $m \equiv 1 \pmod{\gcd(\Omega^)}$ tends to infinity.*

- iii) *For any subset $\Lambda \subset \mathbb{N}$ the series*

$$E^{\Lambda}(z, w) = z Z_{\text{SET}_{\Lambda}}(w, \tilde{\mathcal{A}}_{\Omega^*}(z^2), \tilde{\mathcal{A}}_{\Omega^*}(z^3), \dots)$$

satisfies

$$E^{\Lambda}(\rho + \epsilon, \tilde{\mathcal{A}}_{\Omega^*}(\rho) + \epsilon) < \infty$$

for some $\epsilon > 0$.

In [BFKV11, Prop. 24] the cycle-pointing decomposition was used in order to provide a new method for determining the asymptotic number of free trees. The argument used there can easily be extended to the case of vertex degree restrictions.

Proposition 8.2. *The series $\tilde{\mathcal{F}}_{\Omega}(z)$ and $\tilde{\mathcal{A}}_{\Omega^*}(z)$ both have the same radius of convergence ρ . Moreover, the following statements hold.*

- i) *There is a constant d'_{Ω^*} such that*

$$[z^n] \tilde{\mathcal{F}}_{\Omega}(z) \sim d'_{\Omega^*} \rho^{-n} n^{-5/2}$$

as $n \equiv 2 \pmod{\gcd(\Omega^)}$ tends to infinity.*

- ii) *For any set $\Lambda \subset \mathbb{N}$ the series*

$$F^{\Lambda}(z, w) = \bar{Z}_{\text{SET}_{\Lambda}^{\odot}}(w, \tilde{\mathcal{A}}_{\Omega^*}^{\circ}(z); \tilde{\mathcal{A}}_{\Omega^*}(z^2), \tilde{\mathcal{A}}_{\Omega^*}^{\circ}(z^2); \tilde{\mathcal{A}}_{\Omega^*}(z^3), \tilde{\mathcal{A}}_{\Omega^*}^{\circ}(z^3); \dots)$$

satisfies $F^{\Lambda}(\rho + \epsilon, \tilde{\mathcal{A}}_{\Omega^}(\rho) + \epsilon) < \infty$ for some $\epsilon > 0$.*

- iii) *The power series*

$$\bar{Z}_{\text{SET}_{\{2\}}^{\odot} \odot \mathcal{A}_{\Omega^*}}(z) = \tilde{\mathcal{A}}_{\Omega^*}^{\circ}(z^2)$$

has radius of convergence greater than ρ .

Proof. Let ρ denote the radius of convergence of $\tilde{\mathcal{A}}_{\Omega^*}(z)$. Claim iii) follows from the fact that $\rho < 1$ and the series

$$\tilde{\mathcal{A}}_{\Omega^*}^{\circ}(z) = z \frac{d}{dz} \tilde{\mathcal{A}}_{\Omega^*}(z)$$

also has radius of convergence ρ . We proceed with claim ii). The series $\bar{Z}_{\text{SET}^\oplus_\Lambda}$ is dominated coefficient-wise by the series

$$\bar{Z}_{\text{SET}^\oplus_\Lambda}(s_1, t_1; s_2, t_2; \dots) = \exp\left(\sum_{k=1}^{\infty} s_k/k\right) \sum_{i=2}^{\infty} t_i$$

and hence $F^\Lambda(z, w)$ is dominated by

$$\exp\left(w + \sum_{k=2}^{\infty} \tilde{\mathcal{A}}_{\Omega^*}(z^k)/k\right) \sum_{i=2}^{\infty} \tilde{\mathcal{A}}_{\Omega^*}^\circ(z^i).$$

Since $\rho < 1$ this series is finite for $z = \rho + \epsilon$ and $w = \tilde{\mathcal{A}}_{\Omega^*}(\rho) + \epsilon$ if $\epsilon > 0$ is sufficiently small. In order to prove claim i) we are going to perform a singularity analysis of the series $\tilde{\mathcal{F}}_\Omega^\circ(z)$. The cycle pointing decomposition

$$\mathcal{F}_\Omega^\circ \simeq \mathcal{X}^\circ \star (\text{SET}_\Omega \circ \mathcal{A}_{\Omega^*}) + \text{SET}_{\{2\}}^\oplus \odot \mathcal{A}_{\Omega^*} + (\text{SET}_\Omega^\oplus \odot \mathcal{A}_{\Omega^*}) \star \mathcal{X}$$

yields that the series $\tilde{\mathcal{F}}_\Omega^\circ(z) = z \frac{d}{dz} \tilde{\mathcal{F}}_\Omega(z)$ can be written in the form

$$\tilde{\mathcal{F}}_\Omega^\circ(z) = zh(z, \tilde{\mathcal{A}}_{\Omega^*}(z))$$

with

$$h(z, w) = E^\Omega(z, w) + F^\Omega(z, w) + \tilde{\mathcal{A}}_{\Omega^*}^\circ(z^2)/z.$$

Here we let E^Ω be defined as in Proposition 8.1. Set $d = \gcd(\Omega^*)$. We have that $\tilde{\mathcal{A}}_{\Omega^*}(z)$ satisfies the prerequisites of the type of power series studied in Jason, Stanley and Yeats [BBY06, Thm. 28]: Its dominant singularities (all of square-root type) are given by the rotated points

$$U = \{\omega^k \rho \mid k = 0, \dots, d-1\}$$

with

$$\omega = e^{\frac{2\pi i}{d}}.$$

Moreover

$$\tilde{\mathcal{A}}_{\Omega^*}(\omega z) = \omega \tilde{\mathcal{A}}_{\Omega^*}(z)$$

for all z in a generalized Δ -region with wedges removed at the points of U . We have that $h(z, w)$ is a power series with nonnegative coefficients and by claim i) and ii) and Proposition 8.1 we have

$$h(\tilde{\mathcal{A}}_{\Omega^*}(\rho) + \epsilon, \rho + \epsilon) < \infty$$

for some $\epsilon > 0$. Hence the dominant singularities and their types are driven by the series $\tilde{\mathcal{A}}_{\Omega^*}(z)$. We may apply a standard result for the singularity analysis of functions with multiple dominant singularities [FS09, Thm. VI.5] and obtain that

$$[z^m]h(z, \tilde{\mathcal{A}}_{\Omega^*}(z)) \sim d'_{\Omega^*} m^{-3/2} \rho^{-m}$$

for $m \equiv 1 \pmod{\gcd(\Omega^*)}$ and $d'_{\Omega^*} > 0$ a constant. \square

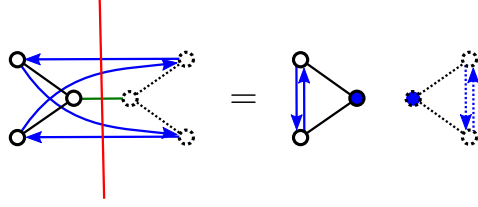


Figure 5. Any unlabelled $\mathcal{E} = \text{SET}_{\{2\}}^{\otimes} \odot \mathcal{A}_{\Omega^*}$ object corresponds to two identical copies of a cycle-pointed Pólya tree.

9. PROOF OF THE MAIN THEOREMS

Throughout this section, let Ω be a set of positive integers containing the number 1 and at least one integer equal or greater than 3. We let \mathcal{F} denote the species of unrooted trees and \mathcal{F}_{Ω} its subspecies of trees with vertex degrees in the set Ω . Analogously, we let \mathcal{A} denote the species of rooted trees and \mathcal{A}_{Ω^*} the subspecies of rooted trees with vertex outdegrees in the shifted set $\Omega^* = \Omega - 1$. In the following we will always assume that n denotes an integer satisfying $n \equiv 2 \pmod{\gcd(\Omega^*)}$ and n large enough such that trees with n vertices and vertex degrees in the set Ω exist. Let ρ denote the radius of convergence of the generating series $\tilde{\mathcal{A}}_{\Omega^*}(z)$.

We let (T_n, τ_n) denote a random cycle-pointed tree drawn uniformly from the unlabelled $\mathcal{F}_{\Omega}^{\circ}$ -objects of size n . As discussed in the preliminaries section, this implies that T_n is the uniform random unlabelled unrooted tree with n vertices and vertex degrees in the set Ω . Moreover, let A_{n-1} a random rooted tree drawn uniformly from the unlabelled \mathcal{A}_{Ω^*} -objects of size $n-1$.

Let $c_{\Omega^*} > 0$ denote the constant such that the uniformly drawn unlabelled rooted tree A_{n-1} satisfies

$$(\mathsf{A}_{n-1}, c_{\Omega^*}(n-1)^{-1/2} d_{\mathsf{A}_{n-1}}) \xrightarrow{(d)} (\mathcal{T}_e, d_{\mathcal{T}_e})$$

with respect to the Hausdorff-Gromov metric. Moreover, let $(\hat{\mathsf{A}}_{\Omega^*}, u)$ denote the infinite rooted tree with

$$(\mathsf{A}_{n-1}, u_{n-1}) \xrightarrow{(d)} (\hat{\mathsf{A}}, u)$$

in the local sense, with respect to a uniformly at random drawn root u_{n-1} of A_{n-1} .

We are going to treat each of the classes \mathcal{S} , \mathcal{E} , and \mathcal{V} separately. The event $(\mathsf{T}_n, \tau_n) \in \mathcal{E}$ is so unlikely, that we will be able to neglect this case.

Lemma 9.1. *There are constants $C, c > 0$, such that*

$$\mathbb{P}((\mathsf{T}_n, \tau_n) \in \mathcal{E}) \leq C \exp(-cn).$$

Proof. The probability for this event is given by the ratio of unlabelled cycle pointed trees of \mathcal{E} with n vertices, and the unlabelled cycle pointed trees in \mathcal{F}_{Ω} with n vertices. Hence

$$\mathbb{P}((\mathsf{T}_n, \tau_n) \in \mathcal{E}) = \frac{[z^n] \tilde{\mathcal{E}}(z)}{[z^n] \tilde{\mathcal{F}}^{\circ}(z)}.$$

By Proposition 8.2, *iii*), the radius of convergence of the ordinary generating series $\tilde{\mathcal{E}}(z)$ is strictly larger than the radius of convergence ρ of $\tilde{\mathcal{F}}^{\circ}(z)$. This yields the claim. \square

Geometrically speaking, this can be explained by the fact that any unlabelled cycle pointed tree from \mathcal{E} corresponds bijectively to a cycle pointed Pólya tree from $\mathcal{A}_{\Omega^*}^{\circ}$ having precisely

half of its size. Compare with Figure 5. The number of such objects is roughly given by $\rho^{n/2}$, while the number of all cycle pointed trees in \mathcal{F}_Ω° is roughly given by ρ^n , which is exponentially larger.

Random trees from \mathcal{V} and \mathcal{S} may be shown separately to admit the same scaling limits, local limits, and tail bounds for the diameter.

Lemma 9.2. *Let S_n be drawn uniformly from the unlabelled*

$$\mathcal{S} = \mathcal{X}^\circ \star (\text{SET}_\Omega \circ \mathcal{A}_{\Omega^*})$$

cycle pointed trees of size n . Then we have

$$(S_n, c_{\Omega^*} n^{-1/2} d_{S_n}) \xrightarrow{(d)} (\mathcal{T}_e, d_{\mathcal{T}_e}),$$

with respect to the Gromov-Hausdorff metric. Moreover, there are constants $C, c > 0$ such that for all n and $x \geq 0$ we it holds that

$$\mathbb{P}(D(T_n) \geq x) \leq C \exp(-cx^2/n).$$

Letting u_n denote a uniformly at random drawn vertex of S_n , it holds that

$$(S_n, u_n) \xrightarrow{(d)} (\hat{\mathcal{A}}_{\Omega^*}, u)$$

in the local sense.

The reason for this is, that each unlabelled $\mathcal{S} = \mathcal{X}^\circ \star (\text{SET}_\Omega \circ \mathcal{A}_{\Omega^*})$ cycle pointed trees corresponds bijectively to a Pólya tree, in which each vertex degree must lie in Ω . That is, the outdegree of the root lies in Ω , and the outdegrees of all remaining vertices lie in Ω^* . Compare with Figure 6. The largest part of difficulties arise in treating the random unlabelled cycle pointed trees of \mathcal{V} , that have a much more complicated bijective decomposition.

Lemma 9.3. *Let V_n be drawn uniformly from the unlabelled*

$$\mathcal{V} = (\text{SET}_\Omega^\circ \odot \mathcal{A}_{\Omega^*}) \star \mathcal{X}$$

objects of size n . Then we have

$$(V_n, c_{\Omega^*} n^{-1/2} d_{V_n}) \xrightarrow{(d)} (\mathcal{T}_e, d_{\mathcal{T}_e}).$$

Moreover, there are constants $C, c > 0$ such that for all $x \geq 0$ and n we have the tail bound

$$\mathbb{P}(D(V_n) \geq x) \leq C \exp(-cx^2/n).$$

Letting u_n denote a uniformly at random drawn vertex of V_n , it holds that

$$(V_n, u_n) \xrightarrow{(d)} (\hat{\mathcal{A}}_{\Omega^*}, u)$$

in the local sense.

The keypoint is that any unlabelled cycle pointed tree from \mathcal{V} corresponds to a Pólya tree A from \mathcal{A}_{Ω^*} where each nonroot vertex must have outdegrees in Ω^* , together with a number K of identical copies of a symmetrically cycle pointed Pólya tree A° from $\mathcal{A}_{\Omega^*}^\circ$, such that the sum of the root degrees of A and the K copies of A° lies in Ω . Compare with Figure 7.

Having these results at hand, we may deduce the scaling limit, the Benjamini-Schramm limit and the tail-bound for the diameter for the random unlabelled tree T_n by building on the corresponding results for the random Pólya tree A_{n-1} .

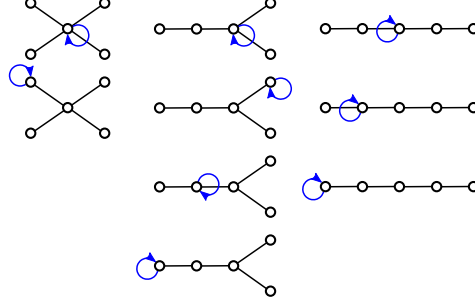


Figure 6. Unlabelled $\mathcal{S} = \mathcal{X}^\circ \star (\text{SET}_{\Omega} \circ \mathcal{A}_{\Omega^*})$ cycle pointed trees correspond to Pólya trees, in which each vertex degree must lie in Ω .

Proof of Theorems 1.1 and 1.2. For the scaling limit, suppose that $f : \mathbb{K} \rightarrow \mathbb{R}$ is a bounded continuous function defined on the space of compact metric spaces equipped with the Gromov-Hausdorff metric. Then Lemmas 9.1, 9.2 and 9.3 imply that

$$\mathbb{E}[f(c_{\Omega^*} n^{-1/2} \mathbb{T}_n)] = \sum_{\mathcal{B} \in \{\mathcal{E}, \mathcal{S}, \mathcal{V}\}} \mathbb{E}[f(c_{\Omega^*} n^{-1/2} \mathbb{T}_n) \mid (\mathbb{T}_n, \tau_n) \in \mathcal{B}] \mathbb{P}((\mathbb{T}_n, \tau_n) \in \mathcal{B}) \rightarrow \mathbb{E}[f(\mathcal{T}_e)].$$

This proves that

$$(\mathbb{T}_n, c_{\Omega^*} n^{-1/2} \mathbb{T}_n) \xrightarrow{(d)} (\mathcal{T}_e, d_{\mathcal{T}_e}).$$

For the tail bound of the diameter, note that it suffices to show such a bound for $\mathbb{P}(D(\mathbb{T}_n) \geq x)$ when $x \leq n$. The lemmas imply that there are constants $C_i, c_i > 0$, for $i = 1, 2, 3$, such that

$$\begin{aligned} \mathbb{P}(D(\mathbb{T}_n) \geq x) &\leq \sum_{\mathcal{B} \in \{\mathcal{E}, \mathcal{S}, \mathcal{V}\}} \mathbb{P}(D(\mathbb{T}_n) \geq x \mid (\mathbb{T}_n, \tau_n) \in \mathcal{B}) \mathbb{P}((\mathbb{T}_n, \tau_n) \in \mathcal{B}) \\ &\leq C_1 \exp(-c_1 n) + \sum_{i=2}^3 C_i \exp(-c_i x^2/n). \end{aligned}$$

As we assumed that $x \leq n$, it holds that

$$\exp(-c_1 n) \leq \exp(-c_1 x^2/n).$$

Hence for a suitable choice of constants $C, c > 0$, it follows that

$$\mathbb{P}(D(\mathbb{T}_n) \geq x) \leq C \exp(-cx^2/n)$$

for all n and $x \geq 0$.

For the local limit, let u_n denote a uniformly at random drawn vertex of the tree \mathbb{T}_n . For any rooted graph G^\bullet , let $V_k(G^\bullet)$ denote the k -neighbourhood of the root in G^\bullet . Then Lemmas

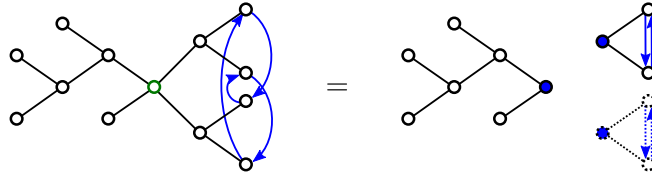


Figure 7. Decomposition of unlabelled $\mathcal{V} = (\text{SET}_{\Omega}^{\oplus} \odot \mathcal{A}_{\Omega^*}) \star \mathcal{X}$ objects into a Pólya tree and a number of identical copies of a symmetrically cycle-pointed Pólya tree.

9.1, 9.2 and 9.3 imply that

$$\begin{aligned} \mathbb{P}(V_k(\mathbf{T}_n, u_n) \simeq G^\bullet) &= \sum_{\mathcal{B} \in \{\mathcal{E}, \mathcal{S}, \mathcal{V}\}} \mathbb{P}(V_k(\mathbf{T}_n, u_n) \simeq G^\bullet \mid (\mathbf{T}_n, \tau_n) \in \mathcal{B}) \mathbb{P}((\mathbf{T}_n, \tau_n) \in \mathcal{B}) \\ &\rightarrow \mathbb{P}(V_k(\hat{\mathbf{A}}_{\Omega^*}, u) \simeq G^\bullet). \end{aligned}$$

Hence

$$(\mathbf{T}_n, u_n) \xrightarrow{(d)} (\hat{\mathbf{A}}, u)$$

in the local sense. \square

Note that this apparently does not work the other way. Convergence of \mathbf{T}_n would not imply that (\mathbf{T}_n, τ_n) conditioned on, say, belonging to \mathcal{V} , converges. And even if we would know that a random tree \mathbf{V}_n from the \mathcal{V} -objects with size n converges, then using our arguments below, we may at best deduce convergence of a \mathcal{A}_{Ω^*} -object having a random size that concentrates around $n - 1$, which does not imply convergence (neither locally, nor in a scaling limit sense) of random Pólya trees conditioned on having precisely $n - 1$ vertices.

It remains to provide proofs for Lemmas 9.2 and 9.3. Although they may seem rather intuitive, one needs to overcome several technical difficulties.

Proof of Lemma 9.2. It holds that

$$\mathcal{S} = \mathcal{X}^\circ \star (\text{SET}_\Omega \circ \mathcal{A}_{\Omega^*}) \simeq \mathcal{X} \cdot (\text{SET}_\Omega \circ \mathcal{A}_{\Omega^*}),$$

hence we do not require cycle pointing techniques in this case. Let (\mathbf{S}_n, σ_n) be drawn uniformly at random from the set $\text{Sym}(\mathcal{S})[n]$. Let π_n denote the corresponding partition. By the discussion in Section 5.4, σ_n induces an automorphism

$$\bar{\sigma}_n : \pi_n \rightarrow \pi_n$$

of the SET_Ω -object. Moreover, let $F_n \subset \pi_n$ denote the fixpoints of $\bar{\sigma}_n$, $f_n = |F_n|$ their number and for each fixpoint $Q \in F_n$ let (\mathbf{A}_Q, σ_Q) denote the corresponding symmetry from $\text{Sym}(\mathcal{A}_{\Omega^*})(Q)$. Let H_n denote the total size of the trees dangling from cycles with length at least 2. We are going to show the following claims.

1) There are constants $C_1 > 0$ and $0 < \gamma < 1$ such that for all n and $x \geq 0$ we have that

$$\mathbb{P}(H_n \geq x) \leq C_1 n^{3/2} \gamma^x$$

and

$$\mathbb{P}(f_n \geq x) \leq C_1 n^{3/2} \gamma^x.$$

2) For any $\delta > 0$ the maximum size $\max_{Q \in F_n} |\mathbf{A}_Q|$ of the trees corresponding to the fixpoints of $\bar{\sigma}_n$ satisfies

$$\mathbb{P}(\max_{Q \in F_n} |\mathbf{A}_Q| \leq n - n^\delta) = o(1).$$

3) There is a constant $C_2 > 0$ such that

$$\mathbb{E}[f_n] \leq C_2$$

for all n .

From claims 1) - 3), we may deduce the tail bound for the diameter as follows. First, it suffices to show such a bound for all $\sqrt{n} \leq x \leq n$. If $D(\mathbf{S}_n) \geq x$, then we have $H_n \geq x/2$ or $\max_{Q \in F_n} H(\mathbf{A}_Q) \geq x/2 - 1$. By 1), we have

$$\mathbb{P}(H_n \geq x/2) \leq C_1 n^{3/2} \gamma^{x/2}$$

and there are constants $C_4, c_4 > 0$ such that

$$C_1 n^{3/2} \gamma^{x/2} \leq C_4 \exp(-c_4 x^2/n)$$

for all n and $\sqrt{n} \leq x \leq n$. Let \mathfrak{E}_n denote the event $\max_Q H(A_Q) \geq x/2 - 1$. It holds that

$$\mathbb{P}(\mathfrak{E}_n) \leq \sum_F \mathbb{P}(F_n = F) \mathbb{P}(\mathfrak{E}_n \mid F_n = F).$$

with F ranging over all subsets of partitions of $[n]$ with $\mathbb{P}(F_n = F) > 0$. By the discussion of symmetries in Section 5.4 we have that given $F_n = F$, the symmetries $(A_Q, \sigma_Q)_{Q \in F}$ are independent and for each $Q \in F$ we have that (A_Q, σ_Q) gets drawn uniformly at random from the set $\text{Sym}(\mathcal{A}_{\Omega^*})[Q]$. That is, A_Q gets drawn uniformly at random from all unlabelled Pólya trees with outdegrees in the set Ω^* . By Inequality (3.2) it follows that there are positive constants C_5, c_5 such that uniformly for all n and x

$$\mathbb{P}(\mathfrak{E}_n \mid F_n = F) \leq C_5 \sum_{Q \in F} \exp(-c_4 x^2/|Q|) \leq |F| C_4 \exp(-c_5 x^2/n).$$

It follows that

$$\mathbb{P}(\mathfrak{E}_n) \leq C_5 \exp(-c_5 x^2/n) \sum_F \mathbb{P}(F_n = F) |F| \leq \mathbb{E}[f_n] C_5 \exp(-c_5 x^2/n).$$

By 3) we have that

$$\mathbb{E}[f_n] \leq C_2$$

for all n . Thus, for some $C_6, c_6 > 0$, it holds that

$$\mathbb{P}(D(S_n) \geq x) \leq C_4 \exp(-c_4 x^2/n) + C_2 C_5 \exp(-c_5 x^2/n) \leq C_6 \exp(-c_6 x^2/n)$$

uniformly for all n and $\sqrt{n} \leq x \leq n$. Thus the claims 1) and 3) imply the tail bound for the diameter.

From claims 1) - 3), we may deduce the convergence towards the CRT as follows. Select one of the partition classes from F_n with maximal size uniformly at random and let X_n denote the corresponding tree. By claim 2) we have

$$\mathbb{P}(|X_n| \leq n - n^{1/4}) = o(1)$$

and thus

$$\mathbb{P}(d_{\text{GH}}(X_n, S_n) \geq n^{1/4}) = o(1).$$

It follows that

$$d_{\text{GH}}(c_{\Omega^*} S_n / \sqrt{n}, c_{\Omega^*} X_n / \sqrt{n}) \xrightarrow{p} 0.$$

Hence it suffices to show

$$c_{\Omega^*} X_n / \sqrt{n} \xrightarrow{(d)} \mathcal{T}_e.$$

Let $f : \mathbb{K} \rightarrow \mathbb{R}$ denote a bounded Lipschitz-continuous function defined on the space $(\mathbb{K}, d_{\text{GH}})$ of isometry classes of compact metric spaces equipped with the Gromov-Hausdorff metric. By claim 2) it follows that

$$\mathbb{E}[f(\frac{c_{\Omega^*}}{\sqrt{n}} X_n)] = o(1) + \sum_{\ell} \mathbb{P}(|X_n| = \ell) \mathbb{E}[f(\frac{c_{\Omega^*}}{\sqrt{n}} X_n) \mid |X_n| = \ell].$$

with the index of the sum ranging over all integers $n - n^{1/4} \leq \ell \leq n$ satisfying $\mathbb{P}(|X_n| = \ell) > 0$, in particular $\ell \equiv 1 \pmod{\gcd(\Omega^*)}$. Since $\ell > n/2$ we have by the discussion of the structure of

symmetries in Section 5.4 that \mathbf{X}_n conditioned $|\mathbf{X}_n| = \ell$ is distributed like a uniformly drawn Pólya tree \mathbf{A}_ℓ of size ℓ with outdegrees in Ω^* . Hence

$$\mathbb{E}[f(\frac{c_{\Omega^*}}{\sqrt{n}}\mathbf{X}_n) \mid |\mathbf{X}_n| = \ell] = \mathbb{E}[f(\frac{c_{\Omega^*}}{\sqrt{n}}\mathbf{A}_\ell)] = \mathbb{E}[f(\frac{c_{\Omega^*}}{\sqrt{\ell}}\mathbf{A}_\ell)] + R_\ell$$

with

$$|R_\ell| \leq C \left| \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{\ell}} \right| \mathbb{E}[D(\mathbf{A}_\ell)]$$

for a fixed constant $C > 0$ that does not depend on ℓ . We have by Inequality (3.2) that

$$\mathbb{E}[D(\mathbf{A}_\ell)] = O(\sqrt{\ell}),$$

hence

$$\sum_{\ell} R_\ell = o(1).$$

By assumption,

$$\mathbb{E}[f(\frac{c_{\Omega^*}}{\sqrt{\ell}}\mathbf{A}_\ell)] \rightarrow \mathbb{E}[f(\mathcal{T}_e)]$$

and hence it follows that

$$\mathbb{E}[c_{\Omega^*}\mathbf{X}_n/\sqrt{n}] \rightarrow \mathbb{E}[f(\mathcal{T}_e)].$$

Thus claim 2) implies that

$$c_{\Omega^*}\mathbf{S}_n/\sqrt{n} \xrightarrow{(d)} \mathcal{T}_e.$$

From claims 1) - 3), we may deduce the local convergence as follows. Let T^\bullet be a fixed rooted tree, and $k \geq 1$ be given. Let u_n be a uniformly at random drawn vertex of \mathbf{S}_n . We need to show that

$$\mathbb{P}(V_k(\mathbf{S}_n, u_n) \simeq T^\bullet) \rightarrow \mathbb{P}(V_k(\hat{\mathbf{A}}, u) \simeq T^\bullet)$$

as n becomes large, with $V_k(\cdot)$ denoting the k -neighbourhood. Note that 2) implies that with probability tending to one it holds that $|\mathbf{X}_n| \geq n - n^{1/4}$ and $u_n \in \mathbf{X}_n$. Let \mathcal{E}_n denote this event. Hence

$$\mathbb{P}(V_k(\mathbf{S}_n, u_n) \simeq T^\bullet) = o(1) + \sum_{n - n^{1/4} \leq \ell \leq n} \mathbb{P}(V_k(\mathbf{S}_n, u_n) \simeq T^\bullet \mid \mathcal{E}_n) \mathbb{P}(\mathcal{E}_n).$$

For any $\ell \geq 1$, the conditioned tree $((\mathbf{X}_n, u_n) \mid \mathcal{E}_n)$ is distributed like a uniformly at random drawn Pólya tree \mathbf{A}_ℓ from \mathcal{A}_{Ω^*} with ℓ vertices, together with a uniformly at random drawn root a_ℓ from \mathbf{A}_ℓ .

As ℓ becomes large, the maximum degree of such a tree that is bounded by $C \log \ell$ for some constant C that does not depend on ℓ . This crude bound is implicit in [PS15] and [Stu15]. In particular, there are at most $C^k \log^k \ell$ vertices with distance at most k of the root. Thus, as ℓ becomes large, the random vertex $(u_n \mid \mathcal{E}_n)$ in the conditioned tree $((\mathbf{X}_n, u_n) \mid \mathcal{E}_n)$ has distance strictly greater than k from the root, and hence

$$(V_k(\mathbf{S}_n, u_n) \mid \mathcal{E}_n) = (V_k(\mathbf{X}_n, u_n) \mid \mathcal{E}_n)$$

with probability tending to one, uniformly for all ℓ for all $n - n^{1/4} \leq \ell \leq n$. Hence

$$\begin{aligned} \mathbb{P}(V_k(\mathbf{S}_n, u_n) \simeq T^\bullet) &= o(1) + \sum_{n - n^{1/4} \leq \ell \leq n} \mathbb{P}(V_k(\mathbf{X}_n, u_n) \simeq T^\bullet \mid \mathcal{E}_n) \mathbb{P}(\mathcal{E}_n) \\ &= o(1) + \sum_{n - n^{1/4} \leq \ell \leq n} \mathbb{P}(V_k(\mathbf{A}_\ell, a_\ell) \simeq T^\bullet) \mathbb{P}(\mathcal{E}_n). \end{aligned}$$

By (4.1) it holds that

$$\mathbb{P}(V_k(\mathbf{A}_\ell, a_\ell) \simeq T^\bullet) \rightarrow \mathbb{P}(V_k(\hat{\mathbf{A}}, u) \simeq T^\bullet)$$

and hence

$$\mathbb{P}(V_k(\mathbf{S}_n, u_n) \simeq T^\bullet) \rightarrow \mathbb{P}(V_k(\hat{\mathbf{A}}, u) \simeq T^\bullet).$$

This proves the local convergence.

It remains to verify claims 1) - 3). The probability generating function of H_n is given by

$$\mathbb{E}[w^{H_n}] = \frac{[z^{n-1}]Z_{\text{SET}_\Omega}(\tilde{\mathcal{A}}_{\Omega^*}(\rho z), \tilde{\mathcal{A}}_{\Omega^*}((\rho w z)^2), \tilde{\mathcal{A}}_{\Omega^*}((\rho w z)^3), \dots)}{[z^{n-1}]Z_{\text{SET}_\Omega}(\tilde{\mathcal{A}}_{\Omega^*}(\rho z), \tilde{\mathcal{A}}_{\Omega^*}((\rho z)^2), \dots)}$$

Since $1 \in \Omega$ we may bound the denominator from below by $[z^{n-1}]\tilde{\mathcal{A}}_{\Omega^*}(\rho z)$ and by Proposition 8.1 we have that

$$[z^{n-1}]\tilde{\mathcal{A}}_{\Omega^*}(\rho z) \sim Cn^{-3/2}$$

for some constant $C > 0$ as $n \equiv 2 \pmod{\gcd(\Omega^*)}$ tends to infinity. Moreover, for all n the polynomial in the indeterminate w in the numerator is dominated coefficient wise by the series

$$Z_{\text{SET}_\Omega}(\tilde{\mathcal{A}}_{\Omega^*}(\rho), \tilde{\mathcal{A}}_{\Omega^*}((\rho w)^2), \dots)$$

which by Proposition 8.1 has radius of convergence strictly greater than 1. In particular we have that

$$\sum_{k \geq x} [w^k]Z_{\text{SET}_\Omega}(\tilde{\mathcal{A}}_{\Omega^*}(\rho), \tilde{\mathcal{A}}_{\Omega^*}((\rho w)^2), \dots) = O(\gamma^x)$$

for some constant $0 < \gamma < 1$. Hence there is a constant C' such that $\mathbb{P}(H_n \geq x) \leq C'n^{3/2}\gamma^x$ for all n and x . The probability generating function for the random number f_n is given by

$$\mathbb{E}[w^{f_n}] = \frac{[z^{n-1}]Z_{\text{SET}_\Omega}(w\tilde{\mathcal{A}}_{\Omega^*}(\rho z), \tilde{\mathcal{A}}_{\Omega^*}((\rho z)^2), \dots)}{[z^{n-1}]Z_{\text{SET}_\Omega}(\tilde{\mathcal{A}}_{\Omega^*}(\rho z), \tilde{\mathcal{A}}_{\Omega^*}((\rho z)^2), \dots)}$$

and the corresponding bound for the event $f_n \geq x$ follows by the same arguments. This proves claim 1).

We proceed with showing claim 2). Let x_n be a given sequence of positive numbers. The event

$$\max_{Q \in F_n} |\mathbf{A}_Q| \leq x_n$$

would imply that

$$n - 1 = H_n + \sum_{Q \in F_n} |A_Q| \leq H_n + x_n f_n.$$

In particular it holds that $H_n \geq (n - 1)/2$ or $f_n \geq (n - 1)/(2x_n)$. Thus, for

$$x_n = cn/\log(n)$$

with $c > 0$ a sufficiently small number, it follows by the tail bounds of claim 1) that

$$\mathbb{P}(\max_{Q \in F_n} |\mathbf{A}_Q| \leq x_n) = o(1).$$

Thus, setting

$$y_n = n - n^{2/3+\epsilon}$$

for any small $\epsilon > 0$, we have that

$$\mathbb{P}(\max_{Q \in F_n} |\mathbf{A}_Q| \leq y_n) = o(1) + \sum_{x_n \leq k \leq y_n} \mathbb{P}(\max_{Q \in F_n} |\mathbf{A}_Q| = k).$$

We can form any unlabelled \mathcal{S} -object by taking an ordered pair of unlabelled \mathcal{A}_{Ω^*} -objects, connecting their roots by an edge, and declaring the root of the first object as the new root of the resulting tree. It follows that the number of unlabelled \mathcal{S} -objects with size n having the property that at least one of the subtrees dangling from the root has size k is bounded by $a_k a_{n-k}$ with $a_i = [z^i] \tilde{\mathcal{A}}_{\Omega^*}(z)$ for all i . Hence

$$\mathbb{P}(\max_{Q \in F_n} |A_Q| = k) \leq a_k a_{n-k} / [z^n] \tilde{S}(z).$$

By Proposition 8.1 we know that $a_i \sim C i^{-3/2} \rho^{-i}$ as $i \equiv 1 \pmod{\gcd(\Omega^*)}$ tends to infinity. Thus

$$\mathbb{P}(\max_{Q \in F_n} |A_Q| \leq y_n) \leq o(1) + C' \sum_{x_n \leq k \leq y_n} (k(n-k)/n)^{-3/2}$$

for some $C' > 0$. Writing $k = n/2 + t$ we obtain $k(n-k)/n = ((n/2)^2 - t^2)/n$ and this quantity strictly decreases as $|t|$ grows. Hence we have $(k(n-k)/n)^{-3/2} \leq n^{2/3+\epsilon}(1+o(1))$ uniformly for all $x_n \leq k \leq y_n$, and thus $\mathbb{P}(\max_{Q \in F_n} |A_Q| \leq y_n) = o(1)$. Setting $z_n = n - n^{\frac{2}{3}(\frac{2}{3}+\epsilon)+\epsilon'}$ for a small $\epsilon' > 0$ we may repeat the same arguments to obtain

$$\begin{aligned} \mathbb{P}(\max_{Q \in F_n} |A_Q| \leq z_n) &\leq o(1) + C' \sum_{y_n \leq k \leq z_n} (k(n-k)/n)^{-3/2} \\ &\leq o(1) + O(1)(z_n - y_n)(n^{\frac{2}{3}(\frac{2}{3}+\epsilon)+\epsilon'})^{-3/2} \end{aligned}$$

and this quantity tends to zero. We may repeat the same argument arbitrarily many times and hence obtain that for any $\delta > 0$ we have that

$$\mathbb{P}(\max_{Q \in F_n} |A_Q| \leq n - n^\delta) = o(1).$$

This proves claim 2).

It remains to prove claim 3), i.e. we have to show that $\mathbb{E}[f_n] = O(1)$. If $\Omega \subset \mathbb{N}$ is bounded, then this is trivial. Otherwise it seems to require some work. We have that

$$\mathbb{E}[f_n] = \frac{[z^{n-1}](s_1 \frac{\partial Z_{\text{SET}\Omega}}{\partial s_1})(\tilde{\mathcal{A}}_{\Omega^*}(z), \tilde{\mathcal{A}}_{\Omega^*}(z^2), \dots)}{[z^{n-1}]Z_{\text{SET}\Omega}(\tilde{\mathcal{A}}_{\Omega^*}(z), \tilde{\mathcal{A}}_{\Omega^*}(z^2), \dots)}.$$

Since $1 \in \Omega$ we have that the denominator is bounded from below by $[z^{n-1}] \tilde{\mathcal{A}}_{\Omega^*}(z)$. By Proposition 8.1 it follows that

$$([z^{n-1}] \tilde{\mathcal{A}}_{\Omega^*}(z))^{-1} = O(n^{3/2} \rho^n).$$

The power series in z in the numerator is bounded coefficient wise by

$$(s_1 \frac{\partial Z_{\text{SET}}}{\partial s_1})(\tilde{\mathcal{A}}_{\Omega^*}(z), \tilde{\mathcal{A}}_{\Omega^*}(z^2), \dots) = \tilde{\mathcal{A}}_{\Omega^*}(z) \exp(\sum_{i=1}^{\infty} \tilde{\mathcal{A}}_{\Omega^*}(z^i)/i) = h(\tilde{\mathcal{A}}_{\Omega^*}(z))g(z)$$

with

$$h(w) = w \exp(w)$$

analytic on \mathbb{C} and

$$g(w) = \exp(\sum_{i \geq 2} \tilde{\mathcal{A}}_{\Omega^*}(z^i)/i)$$

having radius of convergence strictly larger than ρ since $\rho < 1$. By a singularity analysis using results from [BBY06] and [FS09, Thm. VI.5] it follows that

$$[z^{n-1}]h(\tilde{\mathcal{A}}_{\Omega^*}(z))g(z) = O(n^{-3/2}\rho^{-n}).$$

The detailed arguments are identical as in the proof of Proposition 8.2. This concludes the proof. \square

Proof of Lemma 9.3. The proof is analogous to the proof of Lemma 9.2, but with pointed cycle index sums replacing the role of cycle index sums. Let $(V_n, \tau_n, \sigma_n, v_n)$ be a rooted c-symmetry drawn uniformly at random from the set $\text{RSym}(\mathcal{S})[n]$. In particular, V_n is distributed like the uniformly at random chosen unlabelled \mathcal{V} -object with size n . Let π_n denote the corresponding partition. By the discussion in Section 5.4, σ_n induces an automorphism

$$\bar{\sigma}_n : \pi_n \rightarrow \pi_n$$

of the SET_{Ω} -object. Moreover, let $F_n \subset \pi_n$ denote the fixpoints of $\bar{\sigma}_n$, $f_n = |F_n|$ their number and for each fixpoint $Q \in F_n$ let (A_Q, σ_Q) denote the corresponding symmetry from $\text{Sym}(\mathcal{A}_{\Omega^*})(Q)$. Let H_n denote the total size of the trees dangling from cycles with length at least 2. We are going to show the following claims.

- 1) There are constants $C_1 > 0$ and $0 < \gamma < 1$ such that for all n and $x \geq 0$ we have that

$$\mathbb{P}(H_n \geq x) \leq C_1 n^{3/2} \gamma^x$$

and

$$\mathbb{P}(f_n \geq x) \leq C_1 n^{3/2} \gamma^x.$$

- 2) For any $\delta > 0$ the maximum size $\max_{Q \in F_n} |A_Q|$ of the trees corresponding to the fixpoints of $\bar{\sigma}_n$ satisfies

$$\mathbb{P}(\max_{Q \in F_n} |A_Q| \leq n - n^\delta) = o(1).$$

- 3) There is a constant $C_2 > 0$ such that

$$\mathbb{E}[f_n] \leq C_2$$

for all n .

From these claims we may deduce the tail bounds for the diameter, the scaling limit and the local weak limit in an identical manner as in the proof of Lemma 9.2. It remains to verify claims 1)-3). We start with claim 1). The probability generating function of H_n is given by

$$\mathbb{E}[w^{H_n}] = \frac{[z^{n-1}]\bar{Z}_{\text{SET}_{\Omega}^{\oplus}}(\tilde{\mathcal{A}}_{\Omega^*}(\rho z), \tilde{\mathcal{A}}_{\Omega^*}^{\circ}(\rho z); \tilde{\mathcal{A}}_{\Omega^*}((\rho w z)^2), \tilde{\mathcal{A}}_{\Omega^*}^{\circ}((\rho w z)^2); \dots)}{[z^{n-1}]\bar{Z}_{\text{SET}_{\Omega}^{\oplus}}(\tilde{\mathcal{A}}_{\Omega^*}(\rho z), \tilde{\mathcal{A}}_{\Omega^*}^{\circ}(\rho z); \tilde{\mathcal{A}}_{\Omega^*}((\rho z)^2), \tilde{\mathcal{A}}_{\Omega^*}^{\circ}((\rho z)^2); \dots)}.$$

Since $1 \in \Omega$ and there is a number $k \geq 3$ with $k \in \Omega$ it follows that the denominator is bounded from below by

$$[z^{n-1}]z^{k-1}\tilde{\mathcal{A}}_{\Omega^*}(\rho z) = [z^{n-k}]\tilde{\mathcal{A}}_{\Omega^*}(\rho z).$$

We have that

$$n - k \equiv 1 \pmod{\gcd(\Omega^*)}$$

and thus, by Proposition 8.1, we have that

$$[z^{n-k}]\tilde{\mathcal{A}}_{\Omega^*}(\rho z) \sim C n^{-3/2}$$

as $n \equiv 2 \pmod{\gcd(\Omega^*)}$ tends to infinity. The polynomial in the numerator with indeterminate w is bounded coefficient wise by the series

$$\bar{Z}_{\text{SET}_\Omega^\circ}(\tilde{\mathcal{A}}_{\Omega^*}(\rho), \tilde{\mathcal{A}}_{\Omega^*}^\circ(\rho); \tilde{\mathcal{A}}_{\Omega^*}((\rho w)^2), \tilde{\mathcal{A}}_{\Omega^*}^\circ((\rho w)^2); \dots)$$

which does not depend on n and, by Proposition 8.2, has radius of convergence strictly larger than 1. It follows that there is a constant C' such that

$$\mathbb{P}(H_n \geq x) \leq C' n^{3/2} \gamma^x$$

for all n and x . The probability generating function for the random number number f_n is given by

$$\mathbb{E}[w^{f_n}] = \frac{[z^{n-1}] \bar{Z}_{\text{SET}_\Omega^\circ}(w \tilde{\mathcal{A}}_{\Omega^*}(\rho z), w \tilde{\mathcal{A}}_{\Omega^*}^\circ(\rho z); \tilde{\mathcal{A}}_{\Omega^*}((\rho z)^2), \tilde{\mathcal{A}}_{\Omega^*}^\circ((\rho z)^2); \dots)}{[z^{n-1}] \bar{Z}_{\text{SET}_\Omega^\circ}(\tilde{\mathcal{A}}_{\Omega^*}(\rho z), \tilde{\mathcal{A}}_{\Omega^*}^\circ(\rho z); \tilde{\mathcal{A}}_{\Omega^*}((\rho z)^2), \tilde{\mathcal{A}}_{\Omega^*}^\circ((\rho z)^2); \dots)}.$$

and the corresponding bound for the event $f_n \geq x$ follows by the same arguments. This proves claim 1).

We proceed with showing claim 2). Let x_n be a given sequence of positive numbers. The event

$$\max_{Q \in F_n} |A_Q| \leq x_n$$

would imply that

$$n - 1 = H_n + \sum_{Q \in F_n} |A_Q| \leq H_n + x_n f_n.$$

In particular it holds that $H_n \geq (n - 1)/2$ or $f_n \geq (n - 1)/(2x_n)$. Thus, for

$$x_n = cn / \log(n)$$

with $c > 0$ a sufficiently small number, it follows by the tail bounds of claim 1) that

$$\mathbb{P}(\max_{Q \in F_n} |A_Q| \leq x_n) = o(1).$$

Setting

$$y_n = n - n^{2/3+\epsilon}$$

for any small $\epsilon > 0$, we have that

$$\mathbb{P}(\max_{Q \in F_n} |A_Q| \leq y_n) = o(1) + \sum_{x_n \leq k \leq y_n} \mathbb{P}(\max_{Q \in F_n} |A_Q| = k).$$

Any unlabelled \mathcal{V} -object with a tree of size k dangling from the root that does not contain any vertex of the marked cycle can be formed by connecting the roots of an unlabelled \mathcal{A}_{Ω^*} -object of size k and an unlabelled $\text{SET}_\Omega^\circ \odot \mathcal{A}_{\Omega^*}$ object of size $n - k$. By a singularity analysis similar to the proof of claim 3) in Lemma 9.2 we have that the number b_i of unlabelled $\text{SET}_\Omega^\circ \odot \mathcal{A}_{\Omega^*}$ -objects of size i is at most $O(i^{-3/2} \rho^{-i})$. It follows that

$$\mathbb{P}(\max_{Q \in F_n} |A_Q| = k) \leq ([z^k] \tilde{\mathcal{A}}_{\Omega^*}(z)) b_{n-k} / ([z^n] \tilde{\mathcal{V}}(z)) = O((k(n - k)/n)^{-3/2})$$

uniformly for all $x_n \leq k \leq y_n$ and thus

$$\mathbb{P}(\max_{Q \in F_n} |A_Q| \leq y_n) = o(1) + O(1) \sum_{x_n \leq k \leq y_n} (k(n - k)/n)^{-3/2}.$$

In order to finish the proof of claim 2) we may now follow precisely the same arguments as in the proof of claim 2) in Lemma 9.2.

Claim 3) follows by similar arguments as in the proof of claim 3) in Lemma 9.2. This completes the proof. \square

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